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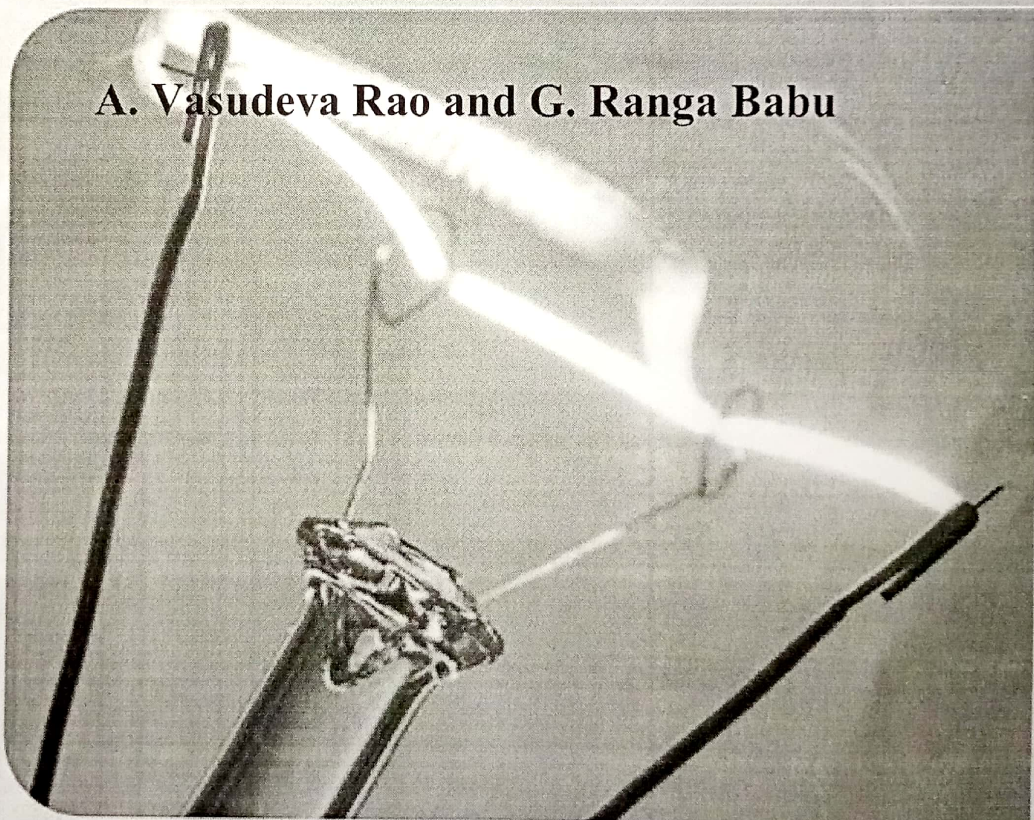

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Research Article

Unbiased Linear Approximate MLEs for the Location and Scale Parameters of the Half-Logistic Distribution Under Type-II Right Censoring

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Abstract: Since the ML method does not yield an explicit estimator for the scale parameter of half-logistic distribution even in complete samples, this paper provide a simple method of deriving a linear estimator, based on a Type-II right censored sample, by making linear approximations to the intractable terms of the likelihood equation of scale parameter. A Monte Carlo simulation study is made to investigate the performance of this new linear approximate MLE (LAMLE) in relation to MLE and Balakrishnan and Wong's[7] AMLE. Based on the LAMLE of scale parameter and the MLE of location parameter, we construct 'unbiased' LAMLEs for location and scale parameters of half-logistic distribution from Type-II right censored samples. When compared these estimators with the corresponding BLUEs based on exact variances, surprisingly, we found that these estimators are just as efficient as the BLUEs. Finally, we present some numerical examples to compare various estimators developed here.

Keywords: Half-logistic distribution; Type-II right censored sample; linear approximate maximum likelihood estimator; unbiased linear approximate maximum likelihood estimators.

INTRODUCTION

Balakrishnan[3] proposed the half-logistic distribution as a life-testing model and demonstrated its' applications. An attraction of this distribution in the context of reliability theory is that it has a monotonically increasing hazard rate for all parameter values, a property shared by relatively few distributions which has support on the positive real half-line. This distribution has also been used successfully to model records. For example, Mbah and Tsokos[14] applied it to environmental and sports records data. A random variable X is said to have a half-logistic distribution with location parameter μ and scale parameter σ if its probability density function (p.d.f.) is given as

$$g(x;\mu,\sigma)=\frac{2e^{-(x-\mu)/\sigma}}{\sigma[1+e^{-(x-\mu)/\sigma}]^2},\ x\geq\mu\geq 0,\ \sigma>0 \qquad [1]$$

And its cumulative distribution function (c.d.f.) and hazard function are

$$G(x;\mu,\sigma)=\frac{1-e^{-(x-\mu)/\sigma}}{1+e^{-(x-\mu)/\sigma}},\ h(x;\mu,\sigma)=\frac{g(x;\mu,\sigma)}{1-\ G(x;\mu,\sigma)}=\frac{1}{\sigma\left[1+e^{-(x-\mu)/\sigma}\right]} \qquad [2]$$

Various methods for estimation of the parameters of the half-logistic distribution, based on both censored and uncensored data, have been discussed by several authors. For example, see Balakrishnan and Puthenpura[6], Balakrishnan and Wong[7&8], Balakrishnan and Chan[5], Adatia[1&2], Rosaiah *et. al*[16], Balakrishnan and Asgharzadeh[4], and David[9]. Acceptance sampling plans based on life test data following half-logistic model for a given consumer's risk are worked out by Kantam and Rosaiah[10]. Estimation of stress-strength reliability where stress, strength variates are assumed to follow half-logistic distribution is discussed by Kantam *et. al*[11]. Consider a life-testing experiment in which n components are put to test simultaneously and their failure times are recorded. Some final observations are censored possibly due to the experimenter terminating the experiment before all the components are failed. Let

$$X_{1:n}\leq X_{r:n}\leq\ldots\leq X_{n-s:n}$$

be an available Type-II right censored sample from the half-logistic distribution with p.d.f. as in Eq. [1], where the largest s observations are censored from the above planned sample of size n. The likelihood function based on the above Type-II censored sample is

$$L=\frac{n!}{s!}\,[1\text{-}G(x_{n-s:n})]^s\prod_{i=1}^{n-s}g(x_{i:n}) \qquad [3]$$

Since, L is monotonically increasing function of μ, the MLE of μ say $\hat{\mu}$ is the smallest order statistic of the sample (see Balakrishnan and Wong[7], p.142). Thus, we have

$$\hat{\mu}=x_{1:n} \qquad [4]$$

By denoting

$$Z_{i:n}=(X_{i:n}-\mu)/\sigma \qquad [5]$$

the likelihood function L can be rewritten as

$$L = \frac{n!}{s!}\sigma^{-(n-s)}\,[1-F(z_{n-s:n})]^{s}\prod_{i=1}^{n-s} f(z_{i:n})$$

where $f(z) = \frac{2e^{-z}}{[1+e^{-z}]^2}$ and $F(z) = \frac{1-e^{-z}}{1+e^{-z}}$ [6]

are respectively the p.d.f. and c.d.f. of standard half-logistic distribution. Now, by realizing that

$$f(z) = \frac{[1-F(z)][1+F(z)]}{2} \qquad [7]$$

the likelihood equation for σ is given by

$$\frac{\partial \log L}{\partial \sigma} = -\frac{1}{2\sigma}\left[2(n-s) - sz_{n-s:n} - sz_{n-s:n}F(z_{n-s:n}) - 2\sum_{i=1}^{n-s} z_{i:n}F(z_{i:n})\right] = 0 \qquad [8]$$

Since, the ML equation [8] does not admit explicit solution for σ even in complete sample, Balakrishnan and Wong[7] derived an approximate maximum likelihood estimator (AMLE) of σ, denoted by $\breve{\sigma}$, and is given by

$$\breve{\sigma} = \left[B + \sqrt{B^2 + 8AC}\right]/4A \qquad [9]$$

where

$$A = n - s,\; B = s(1+\alpha_{n-s})(x_{n-s:n} - x_{1:n}) + 2\sum_{i=1}^{n-s}\alpha_i(x_{i:n} - x_{1:n})$$

$$C = s\beta_{n-s}(x_{n-s:n} - x_{1:n})^2 + 2\sum_{i=1}^{n-s}\beta_i(x_{i:n} - x_{1:n})^2,$$

$$\alpha_i = p_i - \beta_i \log[(1+p_i)/(1-p_i)],\; \beta_i = (1-p_i^2)/2,\; p_i = i/(n+1),\quad 1 \le i \le n-s$$

Balakrishnan and Wong[7]simulated the mean and variance of the AMLE $\breve{\sigma}$ (based on 10000 Monte Carlo runs) for complete and Type-II right censored samples of size up to 20. Using the simulated unbiasing factor, they have constructed unbiased AMLEs of location and scale parameters of half-logistic distribution and have shown them as efficient as the corresponding BLUEs. However, their findings are based on a Monte Carlo simulation study.

In this paper, we derive a new AMLE for scale parameter σ of half-logistic distribution, based on complete and Type-II right censored samples, by suggesting a linear approximation to the intractable terms of the likelihood equation of σ. The basic difference between Balakrishnan and Wong's AMLE and the new AMLE is that the former is a non-linear estimator where as the later is a linear estimator. In the following section, we present this new linear AMLE (LAMLE) along with the mean and variance of the estimator. In order to assess the relative performance of the LAMLE, we compare it with MLE and AMLE based on *simulated* bias and variance. In the next section, we construct unbiased LAMLEs of location and scale parameters and compare them with the corresponding BLUEs and found them almost as efficient as the BLUEs. In the last section, we demonstrate the newly developed estimators by means of some illustrations.

A LINEAR APPROXIMATE MLE OF SCALE PARAMETER

In this section, we propose a new AMLE of scale parameter in the half-logistic distribution, based on the complete and Type-II right censored samples, by expanding the term $z_{i:n}F(z_{i:n})$ in a Taylor Series around the i^{th} population quantile $\xi_i = F^{-1}(p_i) = \log[(1+p_i)/(1-p_i)]$, $p_i = (i-0.3)/(n+0.4)$, and then approximating it by

$$z_{i:n}F(z_{i:n}) \approx \lambda_i + \delta_i z_{i:n}, \qquad 1 \le i \le n-s, \qquad [10]$$

where $\delta_i = F(\xi_i) + \xi_i F'(\xi_i) = p_i + 0.5(1-p_i^2)\log[(1+p_i)/(1-p_i)]$

$$\lambda_i = (F(\xi_i) - \delta_i)\xi_i = (p_i - \delta_i)\log[(1+p_i)/(1-p_i)]$$

Here, it may be noted that Balakrishnan and Wong[7] made linear approximation to the term $F(z_{i:n})$, where as we make linear approximation to the term $z_{i:n}F(z_{i:n})$. Further, it may be noted that they defined ξ_i with $p_i = i/(n+1)$, where as we update ξ_i according to the modified formula of population quantile (see Paul and David[15], ch.6, p.147).

Using the linear approximation [10], the ML equation [8] will be approximated as

$$\frac{\partial \log L}{\partial \sigma} \approx \frac{\partial \log \tilde{L}}{\partial \sigma} = -\frac{1}{2\sigma}\left[2(n-s) - s z_{n-s:n} - s(\lambda_{n-s} + \delta_{n-s} z_{n-s:n}) - 2\sum_{i=1}^{n-s}(\lambda_i + \delta_i z_{i:n})\right] = 0 \qquad [11]$$

Upon using MLE of μ from Eq. [4] and solving Eq. [11] for σ, we get a new AMLE of σ, **denoted by $\tilde{\sigma}$ and is given by**

$$\tilde{\sigma} = \sum_{i=1}^{n-s} m_i x_{i:n} \qquad [12]$$

where $m_i = M_i / D$, *for* $i = 2,\ldots,n-s$ and $m_1 = -\sum_{i=2}^{n-s} m_i$

$$D = 2(n-s) - s\lambda_{n-s} - 2\sum_{i=1}^{n-s}\lambda_i, \quad M_i = \begin{cases} 2\delta_i & \text{if } i < n-s \\ s + (2+s)\delta_{n-s} & \text{if } i = n-s \end{cases}$$

Since, the approximate ML equation [11] is in linear form, we call the above estimator as linear approximate maximum likelihood estimator (LAMLE). From Eq. [12], we may note that $\sum_{i=1}^{n-s} m_i = 0$ that is the linear coefficients of LAMLE $\tilde{\sigma}$ are sum up to zero. From Eq. [4], we may note that MLE $\hat{\mu}$ is also a linear estimator, whose linear coefficients are sum up to 1.

Thus, both MLE $\hat{\mu}$ and LAMLE $\tilde{\sigma}$ are linear estimators and their means, variances and covariance are

$$E(\hat{\mu}) = E(X_{1:n}) = \mu + \sigma\, a_{1:n}\ ,\ E(\tilde{\sigma}) = E\left[\sum_{i=1}^{n-s} m_i(\mu + \sigma Z_{i:n})\right] = \sigma \sum_{i=1}^{n-s} m_i a_{i:n} \qquad [13]$$

$$V(\hat{\mu}) = \sigma^2 b_{11:n},\ V(\tilde{\sigma}) = \sigma^2 \sum_{i=1}^{n-s}\sum_{j=1}^{n-s} m_i m_j b_{ij:n},\ \ Cov(\hat{\mu},\tilde{\sigma}) = \sigma^2 \sum_{j=1}^{n-s} m_j b_{1j:n} \qquad [14]$$

where $a_{i:n} = E(Z_{i:n})$, $b_{ij:n} = Cov(Z_{i:n}, Z_{j:n})$, $1 \le i, j \le n-s$. The means, variances and covariances of order statistics for standard half-logistic distribution are tabulated by Balakrishnan[3] for sample size up to 15 and using them in the above equations we can compute mean/σ, variance/σ^2 and covariance/σ^2 for $\hat{\mu}$ and $\tilde{\sigma}$.

For sample size beyond 15, the means of order statistics from standard half-logistic distribution can be computed using the recurrence formulae provided in Balakrishnan[3]. Since, these means are essential for our work, we compute them for n=16(1)35 and presented in ***Table 1***. When sample size is beyond 35, the recurrence formulae will not work and hence, for large *n*, an *approximate* value of $a_{i:n}$ can be obtained using the following formula.

$$a_{i:n} = E\left[Z_{i:n}\right] = E\left\{F^{-1}\left[F(Z_{i:n})\right]\right\} \approx F^{-1}\left[E(U_{i:n})\right] = F^{-1}\left[\upsilon_{i:n}\right] = \log\left(\frac{1+\upsilon_{i:n}}{1-\upsilon_{i:n}}\right)$$

where $U_{i:n}$ is the i^{th} order statistic based on a sample of size *n* from uniform distribution *U(0,1)* and $\upsilon_{i:n} = E(U_{i:n})$. It is well known that $\upsilon_{i:n} = i/(n+1)$. But, by comparing the values of $a_{i:n}$ obtained using the above equation at n=35 with those of ***Table 1***, we found a better approximation of $a_{i:n}$ can be achieved when we use $\upsilon_{i:n} = i/(n+0.5)$. Hence, for large n, we have

$$a_{i:n} \approx \log\left[(n+0.5+i)/(n+0.5-i)\right] \qquad [15]$$

Using the above means of order statistics, we can compute the means and hence biases of $\hat{\mu}$ and $\tilde{\sigma}$ for any sample size. When sample size is beyond 15, the recurrence formulae provided in Balakrishnan[3], will not work for computation of the variances and covariances of standard order statistics of half-logistic distribution and hence, in this case we may use the asymptotic variance of $\tilde{\sigma}$, which can be obtained by inverting the Fisher information as given below.

$$V(\breve{\sigma}) = \frac{1}{E\left(\dfrac{-\partial^2 \log \breve{L}}{\partial \sigma^2}\right)} = \sigma^2\left[s - n + 0.5s\lambda_{n-s} + \sum_{i=1}^{n-s}\lambda_i + s(1+\delta_{n-s})a_{n-s} + 2\sum_{i=1}^{n-s}\delta_i a_i\right]^{-1} \quad [16]$$

In order to assess the performance of the LAMLE $\breve{\sigma}$, we compare it with the MLE of σ and Balakrishnan and Wong's AMLE $\breve{\breve{\sigma}}$. It may be noted that Balakrishnan and Wong[7] did not compare the AMLE with MLE. We compute MLE of σ by solving the Eq. [8] using Newton-Raphson iterative method and the MLE thus obtained is denoted as $\hat{\sigma}$. It is important to note that to initiate the iterative method; we use the LAMLE as an initial solution of σ. We use the bias and variance for evaluating the relative performances of LAMLE and AMLE as compared with the MLE.

Though, we have explicit formulae for mean and variance of LAMLE, the means and variances of MLE and AMLE are not mathematically tractable. Hence, we have resorted to Monte Carlo simulation to compute the empirical biases and variances of the three estimators based on 6,000 samples of size n=5(5)30, generated from standard half-logistic distribution. We tabulate these simulated biases and variances in ***Table 2*** for complete as well as Type-II right censored samples of size n=5(5)30 with s=0(1)n/2.

Further, we compute the exact bias/σ of LAMLE using Eq. [13] and exact variance/σ^2 of LAMLE using Eq. [14] (for n≤10) and Eq. [16] (for n>10); and shown them in brackets of the table after the corresponding simulated values. ***Table 2*** reveals the following observations.

- The simulated biases and variances of LAMLE are almost agreed with the corresponding exact values shown in brackets.
- Since, the biases of the three estimators are negative, all the estimators of σ are under estimating σ.
- As expected, the bias and variance are gradually decreasing with respect to sample size (n) and are increasing with respect to the quantity of censoring (s).
- Among the three estimators, with respect to bias AMLE is most preferable and LAMLE is least preferable. At the same time, with respect to variance, LAMLE is most preferable and AMLE is least preferable.
- Thus, AMLE is less biased and less efficient than MLE, whereas LAMLE is more biased and more efficient than MLE.
- When sample size is large (n≥30), all the three estimators are almost equally efficient. In other words, MLE, AMLE and LAMLE are asymptotically equivalent.

Remarks:

1. The choice between AMLE and LAMLE depend on the interest of the practitioner with regard to bias and variance.
2. One advantage with LAMLE is that we have explicit formulae for computing bias and variance of the estimate for any sample size, where as in case of AMLE one can get the simulated bias and variance from the tables of Balakrishnan and Wong[7] only up to a sample of size 20.

Table 1: The means of order statistics of standard half-logistic distribution for n=16(1) 35

n	i	$a_{i:n}$
16	1	0.11836
16	2	0.23817
16	3	0.36028
16	4	0.48562
16	5	0.61528
16	6	0.75060
16	7	0.89322
16	8	1.04532
16	9	1.20976
16	10	1.39064
16	11	1.59400
16	12	1.82945
16	13	2.11373
16	14	2.48025
16	15	3.01241
16	16	4.04360
17	1	0.11171
17	2	0.22466
17	3	0.33953
17	4	0.45711
17	5	0.57828
17	6	0.70409
17	7	0.83585
17	8	0.97518
17	9	1.12421
17	10	1.28581
17	11	1.46403
17	12	1.66490
17	13	1.89801
17	14	2.18011
17	15	2.54457
17	16	3.07479
17	17	4.10415
18	1	0.10578
18	2	0.21261
18	3	0.32108
18	4	0.43183
18	5	0.54560
18	6	0.66324
18	7	0.78579
18	8	0.91452
18	9	1.05102
18	10	1.19741
18	11	1.35653
18	12	1.53243
18	13	1.73113
18	14	1.96220
18	15	2.24237
18	16	2.60501
18	17	3.13351
18	18	4.16125
19	1	0.10044
19	2	0.20179
19	3	0.30454
19	4	0.40925
19	5	0.51651
19	6	0.62704
19	7	0.74168
19	8	0.86142
19	9	0.98752
19	10	1.12157
19	11	1.26566
19	12	1.42262
19	13	1.59649
19	14	1.79327
19	15	2.02253
19	16	2.30099
19	17	2.66201
19	18	3.18898
19	19	4.21526
20	1	0.09562
20	2	0.19203
20	3	0.28965
20	4	0.38895
20	5	0.49044
20	6	0.59472
20	7	0.70247
20	8	0.81450
20	9	0.93181
20	10	1.05562
20	11	1.18752
20	12	1.32959
20	13	1.48465
20	14	1.65672
20	15	1.85179
20	16	2.07944
20	17	2.35637
20	18	2.71595
20	19	3.24154
20	20	4.26651
21	1	0.09124
21	2	0.18317
21	3	0.27616
21	4	0.37060
21	5	0.46693
21	6	0.56566
21	7	0.66736
21	8	0.77268
21	9	0.88245
21	10	0.99762
21	11	1.11942
21	12	1.24943
21	13	1.38971
21	14	1.54307
21	15	1.71354
21	16	1.90710
21	17	2.13330
21	18	2.40886
21	19	2.76713
21	20	3.29148
21	21	4.31526
22	1	0.08725
22	2	0.17510
22	3	0.26388
22	4	0.35393
22	5	0.44562
22	6	0.53939
22	7	0.63572
22	8	0.73516
22	9	0.83836
22	10	0.94612
22	11	1.05941
22	12	1.17944
22	13	1.30776
22	14	1.44644
22	15	1.59829
22	16	1.76732
22	17	1.95951
22	18	2.18442
22	19	2.45874
22	20	2.81583
22	21	3.33905
22	22	4.36175
23	1	0.08359
23	2	0.16772
23	3	0.25266
23	4	0.33871
23	5	0.42621
23	6	0.51552
23	7	0.60704
23	8	0.70126
23	9	0.79871
23	10	0.90004
23	11	1.00603
23	12	1.11764
23	13	1.23608
23	14	1.36290
23	15	1.50014
23	16	1.65064
23	17	1.81837
23	18	2.00933
23	19	2.23306
23	20	2.50625
23	21	2.86226
23	22	3.38445
23	23	4.40617
24	1	0.08023
24	2	0.16093
24	3	0.24235
24	4	0.32476
24	5	0.40844
24	6	0.49371
24	7	0.58092
24	8	0.67048
24	9	0.76283
24	10	0.85851
24	11	0.95818
24	12	1.06259
24	13	1.17270
24	14	1.28971
24	15	1.41517
24	16	1.55113
24	17	1.70039
24	18	1.86695
24	19	2.05679
24	20	2.27944
24	21	2.55161
24	22	2.90664
24	23	3.42789
24	24	4.44871
25	1	0.07713
25	2	0.15468
25	3	0.23287
25	4	0.31193
25	5	0.39212
25	6	0.47372
25	7	0.55702
25	8	0.64238
25	9	0.73018
25	10	0.82087
25	11	0.91498
25	12	1.01315
25	13	1.11614
25	14	1.22490
25	15	1.34063
25	16	1.46486
25	17	1.59965
25	18	1.74780
25	19	1.91329
25	20	2.10210
25	21	2.32378
25	22	2.59501
25	23	2.94914
25	24	3.46952
25	25	4.48951
26	1	0.07426
26	2	0.14889
26	3	0.22410
26	4	0.30009
26	5	0.37708
26	6	0.45532
26	7	0.53506
26	8	0.61663
26	9	0.70032
26	10	0.78656
26	11	0.87576
26	12	0.96847
26	13	1.06529
26	14	1.16700
26	15	1.27454
26	16	1.38910
26	17	1.51222
26	18	1.64594
26	19	1.79307
26	20	1.95758
26	21	2.14546
26	22	2.36624
26	23	2.63660
26	24	2.98990
26	25	3.50949
26	26	4.52871
27	1	0.07159
27	2	0.14353
27	3	0.21597
27	4	0.28912
27	5	0.36316
27	6	0.43832
27	7	0.51481
27	8	0.59293
27	9	0.67291
27	10	0.75516
27	11	0.83995
27	12	0.92784
27	13	1.01925
27	14	1.11486
27	15	1.21541
27	16	1.32184
27	17	1.43534
27	18	1.55744
27	19	1.69019
27	20	1.83638
27	21	2.00000
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27	24	2.67654
27	25	3.02907
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27	27	4.56643
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28	5	0.35024
28	6	0.42258
28	7	0.49605
28	8	0.57106
28	9	0.64761
28	10	0.72630
28	11	0.80710
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28	14	1.06762
28	15	1.16211
28	16	1.26160
28	17	1.36702
28	18	1.47954
28	19	1.60072
28	20	1.73257
28	21	1.87790
28	22	2.04070
28	23	2.22692
28	24	2.44611
28	25	2.71494
28	26	3.06677
28	27	3.58493
28	28	4.60278
29	1	0.06680
29	2	0.13388
29	3	0.20137
29	4	0.26944
29	5	0.33822
29	6	0.40795
29	7	0.47862
29	8	0.55083
29	9	0.62416
29	10	0.69972
29	11	0.77680
29	12	0.85670
29	13	0.93893
29	14	1.02459
29	15	1.11373
29	16	1.20725
29	17	1.30576
29	18	1.41025
29	19	1.52189
29	20	1.64221
29	21	1.77324
29	22	1.91778
29	23	2.07981
29	24	2.26530
29	25	2.48378
29	26	2.75193
29	27	3.10310
29	28	3.62063
29	29	4.63786
30	1	0.06464
30	2	0.12953
30	3	0.19479
30	4	0.26059
30	5	0.32699
30	6	0.39436
30	7	0.46234
30	8	0.53214
30	9	0.60226
30	10	0.67528
30	11	0.74861
30	12	0.82549
30	13	0.90351
30	14	0.98524
30	15	1.06955
30	16	1.15792
30	17	1.25042
30	18	1.34808
30	19	1.45170
30	20	1.56252
30	21	1.68205
30	22	1.81232
30	23	1.95613
30	24	2.11745
30	25	2.30226
30	26	2.52008
30	27	2.78760
30	28	3.13815
30	29	3.65509
30	30	4.67175
31	1	0.06261
31	2	0.12545
31	3	0.18863
31	4	0.25231

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Table 1 (continued)

n i $a_{i:n}$	n i $a_{i:n}$	n i $a_{i:n}$	n i $a_{i:n}$	n i $a_{i:n}$	n i $a_{i:n}$
31 5 0.31646	32 1 0.06071	32 28 2.58897	33 22 1.56670	34 15 0.91071	35 8 0.48379
31 6 0.38173	32 2 0.12163	32 29 2.85534	33 23 1.67539	34 16 1.00711	35 9 0.46557
31 7 0.44699	32 3 0.18285	32 30 3.20478	33 24 1.79288	34 17 1.06793	35 10 0.64248
31 8 0.51498	32 4 0.24457	32 31 3.72064	33 25 1.92118	34 18 1.15338	35 11 0.55249
31 9 0.58147	32 5 0.30653	32 32 4.73626	33 26 2.06309	34 19 1.23108	35 12 0.78733
31 10 0.65307	32 6 0.37007		33 27 2.22258	34 20 1.31766	35 13 0.67930
31 11 0.72193	32 7 0.43224	33 1 0.05892	33 28 2.40560	34 21 1.40662	35 14 0.89724
31 12 0.79711	32 8 0.49963	33 2 0.11803	33 29 2.62171	34 22 1.50151	35 15 0.84505
31 13 0.87042	32 9 0.56100	33 3 0.17740	33 30 2.88756	34 23 1.60226	35 16 0.99825
31 14 0.94933	32 10 0.63377	33 4 0.23733	33 31 3.23651	34 24 1.71037	35 17 1.01763
31 15 1.02885	32 11 0.69554	33 5 0.29707	33 32 3.75188	34 25 1.82726	35 18 1.12118
31 16 1.11296	32 12 0.77229	33 6 0.35953	33 33 4.76702	34 26 1.95499	35 19 1.18379
31 17 1.20007	32 13 0.83847	33 7 0.41750		34 27 2.09635	35 20 1.27091
31 18 1.29189	32 14 0.91712	33 8 0.48702	34 1 0.05723	34 28 2.25530	35 21 1.35273
31 19 1.38866	32 15 0.99074	33 9 0.53904	34 2 0.11464	34 29 2.43781	35 22 1.44255
31 20 1.49152	32 16 1.07206	33 10 0.61957	34 3 0.17225	34 30 2.65342	35 23 1.53635
31 21 1.60157	32 17 1.15385	33 11 0.66641	34 4 0.23059	34 31 2.91878	35 24 1.63665
31 22 1.72037	32 18 1.24085	33 12 0.75381	34 5 0.28785	34 32 3.26725	35 25 1.74415
31 23 1.84993	32 19 1.33159	33 13 0.80465	34 6 0.35056	34 33 3.78217	35 26 1.86050
31 24 1.99306	32 20 1.42771	33 14 0.89051	34 7 0.40139	34 34 4.79687	35 27 1.98770
31 25 2.15373	32 21 1.52981	33 15 0.95324	34 8 0.47963		35 28 2.12854
31 26 2.33791	32 22 1.63916	33 16 1.03573	34 9 0.51106	35 1 0.05563	35 29 2.28699
31 27 2.55512	32 23 1.75728	33 17 1.11065	34 10 0.61677	35 2 0.11144	35 30 2.46902
31 28 2.82204	32 24 1.88619	33 18 1.19452	34 11 0.62630	35 3 0.16736	35 31 2.68415
31 29 3.17202	32 25 2.02869	33 19 1.27947	34 12 0.75029	35 4 0.22442	35 32 2.94906
31 30 3.68840	32 26 2.18874	33 20 1.36999	34 13 0.76025	35 5 0.27844	35 33 3.29708
31 31 4.70452	32 27 2.37233	33 21 1.46523	34 14 0.87637	35 6 0.34431	35 34 3.81157
				35 7 0.38079	35 35 4.82585

Table 2: Performance of AMLE and LAMLE of Scale (σ) parameter of half-logistic distribution compared with MLE based on simulated bias and variance from complete and Type II right censored samples of size n=5(5)30, s=0(1)n/2

n	*s*	Bias/σ			Variance/σ²			Efficiency	
		MLE	AMLE	LAMLE (exact)	MLE	AMLE	LAMLE (exact)	AMLE	LAMLE
5	0	-0.2068	-0.1816	-0.2244 (-0.2245)	0.1200	0.1289	0.1147 (0.1156)	93.09	104.56
5	1	-0.2662	-0.2589	-0.2854 (-0.2833)	0.1380	0.1409	0.1309 (0.1314)	97.97	105.42
5	2	-0.3593	-0.3572	-0.3753 (-0.3740)	0.1642	0.1653	0.1561 (0.1552)	99.32	105.18
10	0	-0.1084	-0.0917	-0.1171 (-0.1171)	0.0663	0.0692	0.0650 (0.0642)	95.81	101.99
10	1	-0.1205	-0.1119	-0.1311 (-0.1312)	0.0719	0.0734	0.0701 (0.0692)	97.88	102.51
10	2	-0.1369	-0.1320	-0.1480 (-0.1482)	0.0790	0.0800	0.0770 (0.0761)	98.79	102.68
10	3	-0.1623	-0.1597	-0.1729 (-0.1692)	0.0877	0.0882	0.0855 (0.0854)	99.36	102.52
10	4	-0.1892	-0.1877	-0.1992 (-0.1961)	0.1022	0.1026	0.0995 (0.0979)	99.60	102.66
10	5	-0.2262	-0.2254	-0.2351 (-0.2325)	0.1191	0.1194	0.1163 (0.1147)	99.79	102.39
15	0	-0.0759	-0.0641	-0.0816 (-0.0794)	0.0448	0.0461	0.0443 (0.0471)	97.17	101.20
15	1	-0.0809	-0.0735	-0.0877 (-0.0857)	0.0473	0.0481	0.0466 (0.0502)	98.28	101.46
15	2	-0.0877	-0.0828	-0.0952 (-0.0927)	0.0502	0.0508	0.0494 (0.0540)	98.83	101.58
15	3	-0.0973	-0.0940	-0.1049 (-0.1007)	0.0535	0.0539	0.0527 (0.0589)	99.26	101.62
15	4	-0.1078	-0.1055	-0.1153 (-0.1098)	0.0593	0.0596	0.0583 (0.0650)	99.46	101.67
15	5	-0.1213	-0.1198	-0.1285 (-0.1205)	0.0649	0.0652	0.0639 (0.0727)	99.64	101.63
15	6	-0.1340	-0.1329	-0.1408 (-0.1333)	0.0720	0.0722	0.0709 (0.0826)	99.76	101.58
15	7	-0.1478	-0.1471	-0.1543 (-0.1489)	0.0802	0.0804	0.0790 (0.0955)	99.82	101.57
20	0	-0.0582	-0.0491	-0.0623 (-0.0601)	0.0339	0.0346	0.0336 (0.0352)	97.94	100.82
20	1	-0.0613	-0.0551	-0.0662 (-0.0637)	0.0351	0.0356	0.0348 (0.0368)	98.72	100.99
20	2	-0.0651	-0.0606	-0.0705 (-0.0675)	0.0371	0.0374	0.0367 (0.0388)	99.07	101.08
20	3	-0.0704	-0.0671	-0.0760 (-0.0717)	0.0391	0.0394	0.0386 (0.0411)	99.29	101.16
20	4	-0.0745	-0.0720	-0.0802 (-0.0763)	0.0411	0.0414	0.0407 (0.0438)	99.48	101.20
20	5	-0.0797	-0.0778	-0.0854 (-0.0814)	0.0445	0.0447	0.0440 (0.0471)	99.58	101.22
20	6	-0.0864	-0.0849	-0.0920 (-0.0871)	0.0474	0.0476	0.0468 (0.0509)	99.67	101.22

Table 2 (continued)

n	s	Bias/σ MLE	AMLE	LAMLE (exact)	Variance/σ² MLE	AMLE	LAMLE (exact)	Efficiency AMLE	LAMLE
20	7	-0.0933	-0.0922	-0.0988 (-0.0936)	0.0512	0.0513	0.0505 (0.0555)	99.74	101.23
20	8	-0.1021	-0.1013	-0.1073 (-0.1010)	0.0552	0.0553	0.0545 (0.0611)	99.82	101.18
20	9	-0.1115	-0.1109	-0.1166 (-0.1097)	0.0600	0.0601	0.0593 (0.0678)	99.85	101.15
20	10	-0.1219	-0.1214	-0.1266 (-0.1198)	0.0657	0.0657	0.0649 (0.0762)	99.90	101.11
25	0	-0.0467	-0.0395	-0.0499 (-0.0484)	0.0273	0.0278	0.0271 (0.0281)	98.30	100.56
25	1	-0.0481	-0.0429	-0.0519 (-0.0507)	0.0280	0.0283	0.0278 (0.0291)	98.80	100.69
25	2	-0.0507	-0.0467	-0.0548 (-0.0531)	0.0291	0.0294	0.0289 (0.0303)	99.13	100.78
25	3	-0.0533	-0.0502	-0.0577 (-0.0557)	0.0302	0.0304	0.0299 (0.0316)	99.31	100.84
25	4	-0.0573	-0.0548	-0.0618 (-0.0584)	0.0313	0.0315	0.0311 (0.0331)	99.50	100.85
25	5	-0.0604	-0.0584	-0.0650 (-0.0614)	0.0328	0.0329	0.0325 (0.0349)	99.62	100.92
25	6	-0.0639	-0.0623	-0.0685 (-0.0647)	0.0345	0.0346	0.0342 (0.0369)	99.67	100.92
25	7	-0.0673	-0.0660	-0.0718 (-0.0682)	0.0364	0.0365	0.0360 (0.0392)	99.72	100.94
25	8	-0.0704	-0.0694	-0.0749 (-0.0721)	0.0387	0.0388	0.0383 (0.0418)	99.78	100.97
25	9	-0.0747	-0.0739	-0.0792 (-0.0765)	0.0415	0.0415	0.0411 (0.0449)	99.79	100.98
25	10	-0.0812	-0.0805	-0.0854 (-0.0814)	0.0443	0.0444	0.0439 (0.0484)	99.86	100.91
25	11	-0.0876	-0.0871	-0.0918 (-0.0868)	0.0473	0.0473	0.0468 (0.0526)	99.89	100.89
25	12	-0.0934	-0.0930	-0.0974 (-0.0931)	0.0509	0.0510	0.0505 (0.0575)	99.91	100.89
30	0	-0.0388	-0.0329	-0.0415 (-0.0405)	0.0229	0.0232	0.0228 (0.0234)	98.62	100.47
30	1	-0.0402	-0.0357	-0.0433 (-0.0421)	0.0234	0.0236	0.0233 (0.0241)	99.04	100.52
30	2	-0.0418	-0.0382	-0.0452 (-0.0437)	0.0240	0.0242	0.0239 (0.0248)	99.31	100.61
30	3	-0.0431	-0.0402	-0.0467 (-0.0455)	0.0248	0.0249	0.0246 (0.0257)	99.47	100.69
30	4	-0.0449	-0.0425	-0.0487 (-0.0474)	0.0258	0.0259	0.0256 (0.0267)	99.53	100.73
30	5	-0.0471	-0.0451	-0.0509 (-0.0493)	0.0266	0.0267	0.0264 (0.0277)	99.60	100.75
30	6	-0.0495	-0.0479	-0.0534 (-0.0514)	0.0273	0.0273	0.0270 (0.0290)	99.67	100.78
30	7	-0.0522	-0.0508	-0.0561 (-0.0537)	0.0286	0.0287	0.0284 (0.0303)	99.70	100.85
30	8	-0.0547	-0.0535	-0.0586 (-0.0561)	0.0302	0.0302	0.0299 (0.0318)	99.73	100.83
30	9	-0.0573	-0.0563	-0.0612 (-0.0587)	0.0318	0.0319	0.0315 (0.0336)	99.74	100.87
30	10	-0.0607	-0.0599	-0.0645 (-0.0616)	0.0334	0.0335	0.0331 (0.0355)	99.80	100.85
30	11	-0.0636	-0.0629	-0.0673 (-0.0647)	0.0353	0.0353	0.0350 (0.0377)	99.84	100.83
30	12	-0.0679	-0.0674	-0.0715 (-0.0681)	0.0375	0.0375	0.0372 (0.0401)	99.88	100.80
30	13	-0.0716	-0.0711	-0.0751 (-0.0719)	0.0397	0.0397	0.0394 (0.0429)	99.89	100.79
30	14	-0.0759	-0.0755	-0.0793 (-0.0761)	0.0421	0.0421	0.0418 (0.0462)	99.92	100.75
30	15	-0.0808	-0.0805	-0.0841 (-0.0809)	0.0447	0.0448	0.0444 (0.0499)	99.93	100.74

CONSTRUCTION OF UNBIASED LAMLES FOR LOCATION AND SCALE PARAMETERS AND COMPARISON WITH THE BLUES

It is well known that among the class of all linear unbiased estimators, the best linear unbiased estimators (BLUEs) are the most efficient estimators since they have least variance. Lloyd[13] obtained explicit formulae for the BLUEs of location and scale parameters in any location-scale family distribution based on a given Type-II censored sample. The evaluation of the linear coefficients of the BLUEs requires the means, variances and covariances of order statistics obtained from the corresponding standardized population. The coefficients of the BLUEs of location and scale parameters of half-logistic distribution for sample size n=2(1)15 are tabulated by Balakrishnan and Puthenpura[6] and Balakrishnan and Wong[8] in complete samples and Type-II censored samples respectively. Adatia[1] constructed approximate BLUEs of location and scale parameters of half-logistic distribution based on an optimally selected few order statistics for fairly large doubly censored samples. Though the LAMLE $\tilde{\sigma}$ constructed in the above section is as efficient as MLE $\hat{\sigma}$, it is biased than MLE. Now, in this section based on MLE $\hat{\mu}$ and LAMLE $\tilde{\sigma}$, we construct unbiased LAMLEs of μ and σ; and we compare them with the corresponding BLUEs. From Eq. [13], we have

$$E\left(\tilde{\sigma}\Big/\sum_{i=1}^{n-s} m_i a_{i:n}\right) = \sigma \text{ and } E\left(\hat{\mu} - \tilde{\sigma} a_{1:n}\Big/\sum_{i=1}^{n-s} m_i a_{i:n}\right) = \mu \qquad [17]$$

Thus, the *unbiased* LAMLEs of σ and μ, denoted by σ* and μ* respectively, are

$$\sigma^* = \sum_{i=1}^{n-s} m_i x_{i:n} \Big/ \sum_{i=1}^{n-s} m_i a_{i:n} \text{ and } \mu^* = x_{1:n} - a_{1:n}\sigma^* \qquad [18]$$

The required $a_{i:n}$ s can be borrowed from Balakrishnan[3] for n≤15 and can be obtained from ***Table 1*** for 16≤n≤35. When n>35, one can compute $a_{i:n}$ directly from Eq. [15]. The variances and covariance of σ* and μ* are

$$\text{Var}(\sigma^*) = \sigma^2 \sum_{i=1}^{n-s}\sum_{j=1}^{n-s} m_i m_j b_{ij:n} \Big/ \left(\sum_{i=1}^{n-s} m_i a_{i:n}\right)^2$$

$$Var(\mu^*) = \sigma^2 \left[b_{1i:n} + a_{1:n}^2 Var(\sigma^*)/\sigma^2 - 2a_{1:n} \sum_{i=1}^{n-s} m_i b_{1i:n} \Big/ \sum_{i=1}^{n-s} m_i a_{i:n} \right]$$

$$Cov(\mu^*, \sigma^*) = \sigma^2 \left[\sum_{i=1}^{n-s} m_i b_{1i:n} \Big/ \sum_{i=1}^{n-s} m_i a_{i:n} - a_{1:n} Var(\sigma^*)/\sigma^2 \right] \qquad [19]$$

The required $b_{ij:n}$ s can be borrowed from Balakrishnan[3] for n≤15.

Since, μ* and σ* are linear unbiased estimators, we compare them with the corresponding BLUEs, denoted by μ** and σ**, based on the exact variances of the estimators. In ***Table 3***, we present the variances, covariance and the efficiencies of μ* and σ* in relation to μ** and σ** respectively, from complete and Type-II right censored samples of size n=3(1)10 and s=0(1)n-2. Looking upon the table, we observe that

- The variances and covariance of the unbiased LAMLEs are almost coinciding with the corresponding variances and covariance of the BLUEs.
- In both complete as well as Type-II right censored samples (even in heavy censoring) μ* and σ* are performing with more than 99% efficiency when compared with the corresponding BLUEs.

The above observations conclude that unbiased LAMLEs are just as efficient as the corresponding BLUEs in both complete and Type-II right censored samples irrespective of its sample size and number of observations censored. It is well known that, the computation of linear coefficients of BLUEs is a very difficult task especially in large samples, since it requires the inversion of the matrix of variances and covariances of standard order statistics. Moreover, when n>15, as explained in the above section, the variances and covariances of standard order statistics of half-logistic distribution are not available. Therefore, when n>15, it is difficult to compute the BLUEs and in this case we may use unbiased LAMLEs as alternative estimators, as they are proved just as efficient as BLUEs.

One advantage with unbiased LAMLEs is that they do not require the variances and covariances of standard order statistics for their computation. Moreover, no separate calculations are needed for the computation of μ* and it can be directly obtained from σ* using Eq. [18]. The variances and covariance of μ* and σ* may be obtained from ***Table 3*** for sample size n=3(1)10 and s=0(1)n-2.

When n>10, we simulate them based on 6000 samples and presented in ***Table 4*** for n=11(1)50 and s=0(1)n/2. These are required to compute the standard errors of the estimators.

Remark: Since, there are no explicit formulae for the variances of Balakrishnan and Wong's unbiased AMLEs; we could not compare them with the BLUEs and unbiased LAMLEs. *Even otherwise, since unbiased AMLEs are non-linear, they are not comparable with BLUEs in the sense that we can find an unbiased non-linear estimator which may be more efficient than the BLUE, but we cannot find an unbiased linear estimator which is more efficient than the BLUE.*

Table 3: Comparison of the Unbiased LAMLEs of location and scale parameters of half-logistic distribution with the corresponding BLUEs based on exact variances of the estimators from complete and Type-II right censored samples of size n=2(1)10.

n	s	BLUEs			Unbiased LAMLEs				
		$V(\mu^{**})$	$V(\sigma^{**})$	$COV(\mu^{**},\sigma^{**})$	$V(\mu^{*})$	$V(\sigma^{*})$	$COV(\mu^{*},\sigma^{*})$	$Eff(\mu^{*})$	$Eff(\sigma^{*})$
3	0	0.3776	0.3968	-0.2468	0.3778	0.3968	-0.2467	99.96	99.99
	1	0.5130	0.8130	-0.4842	0.5130	0.8130	-0.4842	100.00	100.00
4	0	0.2057	0.2601	-0.1314	0.2061	0.2601	-0.1313	99.81	99.99
	1	0.2344	0.3934	-0.1932	0.2347	0.3935	-0.1933	99.88	99.99
	2	0.3232	0.8422	-0.3928	0.3232	0.8422	-0.3928	100.00	100.00
5	0	0.1311	0.1921	-0.0820	0.1313	0.1921	-0.0819	99.84	99.99
	1	0.1410	0.2559	-0.1071	0.1411	0.2559	-0.1071	99.93	100.00
	2	0.1603	0.3960	-0.1591	0.1603	0.3960	-0.1591	99.98	100.00
	3	0.2162	0.8383	-0.3163	0.2162	0.8383	-0.3163	100.00	100.00
6	0	0.0916	0.1521	-0.0565	0.0918	0.1521	-0.0564	99.80	99.99
	1	0.0960	0.1891	-0.0692	0.0961	0.1891	-0.0692	99.89	100.00
	2	0.1032	0.2571	-0.0913	0.1033	0.2571	-0.0913	99.95	100.00
	3	0.1171	0.4013	-0.1361	0.1171	0.4013	-0.1361	99.99	100.00
	4	0.1576	0.8503	-0.2709	0.1576	0.8503	-0.2709	100.00	100.00
7	0	0.0680	0.1257	-0.0414	0.0681	0.1257	-0.0414	99.78	100.00
	1	0.0702	0.1497	-0.0488	0.0703	0.1497	-0.0488	99.86	100.00
	2	0.0736	0.1893	-0.0603	0.0736	0.1893	-0.0603	99.92	100.00
	3	0.0790	0.2597	-0.0798	0.0790	0.2597	-0.0798	99.97	100.00
	4	0.0895	0.4068	-0.1191	0.0895	0.4068	-0.1191	99.99	100.00
	5	0.1202	0.8609	-0.2372	0.1202	0.8609	-0.2372	100.00	100.00
8	0	0.0525	0.1070	-0.0317	0.0527	0.1070	-0.0317	99.76	100.00
8	1	0.0538	0.1237	-0.0364	0.0539	0.1237	-0.0364	99.84	100.00
8	2	0.0556	0.1495	-0.0432	0.0557	0.1495	-0.0432	99.90	100.00
8	3	0.0582	0.1907	-0.0536	0.0583	0.1907	-0.0536	99.95	100.00
8	4	0.0624	0.2628	-0.0710	0.0625	0.2628	-0.0710	99.98	100.00
8	5	0.0706	0.4119	-0.1060	0.0706	0.4119	-0.1060	100.00	100.00
8	6	0.0947	0.8698	-0.2110	0.0947	0.8698	-0.2110	100.00	100.00
9	0	0.0419	0.0931	-0.0251	0.0420	0.0931	-0.0251	99.75	100.00
9	1	0.0427	0.1054	-0.0282	0.0428	0.1054	-0.0282	99.82	100.00
9	2	0.0437	0.1234	-0.0326	0.0438	0.1234	-0.0326	99.88	100.00
9	3	0.0452	0.1503	-0.0388	0.0452	0.1503	-0.0388	99.93	100.00
9	4	0.0473	0.1926	-0.0482	0.0473	0.1926	-0.0482	99.96	100.00
9	5	0.0507	0.2659	-0.0640	0.0507	0.2659	-0.0640	99.99	100.00
9	6	0.0573	0.4166	-0.0955	0.0573	0.4166	-0.0955	100.00	100.00
9	7	0.0765	0.8735	-0.1893	0.0765	0.8735	-0.1893	100.00	100.00
10	0	0.0342	0.0824	-0.0204	0.0343	0.0824	-0.0204	99.74	100.00
10	1	0.0348	0.0917	-0.0226	0.0348	0.0917	-0.0226	99.80	100.00
10	2	0.0354	0.1049	-0.0256	0.0355	0.1049	-0.0256	99.86	100.00
10	3	0.0363	0.1238	-0.0296	0.0363	0.1238	-0.0296	99.91	100.00
10	4	0.0374	0.1514	-0.0353	0.0375	0.1515	-0.0353	99.94	100.00
10	5	0.0392	0.1946	-0.0439	0.0392	0.1946	-0.0439	99.97	100.00
10	6	0.0419	0.2689	-0.0582	0.0419	0.2689	-0.0582	99.99	100.00
10	7	0.0474	0.4211	-0.0870	0.0474	0.4211	-0.0870	100.00	100.00
10	8	0.0634	0.8856	-0.1734	0.0634	0.8856	-0.1734	100.00	100.00

Table 4: The variances and covariance of μ^* and σ^* in half-logistic distribution for n=11(1)50 and s=0(1)n/2 (based on 6000 simulations)

n	s	$\frac{V(\mu^*)}{\sigma^2}$	$\frac{V(\sigma^*)}{\sigma^2}$	$\frac{Cov(\mu^*,\sigma^*)}{\sigma^2}$
11	0	0.0279	0.0739	-0.0167
11	1	0.0282	0.0815	-0.0182
11	2	0.0287	0.0909	-0.0205
11	3	0.0293	0.1046	-0.0232
11	4	0.0298	0.1237	-0.0266
11	5	0.0309	0.1531	-0.0323
12	0	0.0238	0.0661	-0.0143
12	1	0.0240	0.0725	-0.0155
12	2	0.0243	0.0792	-0.0169
12	3	0.0247	0.0899	-0.0190
12	4	0.0251	0.1038	-0.0214
12	5	0.0257	0.1272	-0.0252
12	6	0.0264	0.1558	-0.0296
13	0	0.0201	0.0596	-0.0121
13	1	0.0203	0.0642	-0.0130
13	2	0.0204	0.0703	-0.0139
13	3	0.0207	0.0785	-0.0153
13	4	0.0209	0.0896	-0.0168
13	5	0.0213	0.1045	-0.0193
13	6	0.0216	0.1244	-0.0220
14	0	0.0178	0.0555	-0.0101
14	1	0.0179	0.0593	-0.0109
14	2	0.0180	0.0638	-0.0116
14	3	0.0182	0.0704	-0.0127
14	4	0.0184	0.0794	-0.0140
14	5	0.0186	0.0911	-0.0158
14	6	0.0189	0.1062	-0.0177
14	7	0.0193	0.1248	-0.0204
15	0	0.0153	0.0521	-0.0101
15	1	0.0154	0.0553	-0.0106
15	2	0.0155	0.0595	-0.0114
15	3	0.0157	0.0643	-0.0122
15	4	0.0158	0.0727	-0.0134
15	5	0.0160	0.0817	-0.0145
15	6	0.0161	0.0940	-0.0159
15	7	0.0164	0.1087	-0.0179
16	0	0.0134	0.0479	-0.0083
16	1	0.0135	0.0508	-0.0089
16	2	0.0136	0.0545	-0.0095
16	3	0.0137	0.0592	-0.0103
16	4	0.0138	0.0660	-0.0113
16	5	0.0139	0.0738	-0.0122
16	6	0.0141	0.0832	-0.0134
16	7	0.0143	0.0944	-0.0149
16	8	0.0145	0.1101	-0.0168
17	0	0.0126	0.0445	-0.0078
17	1	0.0126	0.0469	-0.0081
17	2	0.0127	0.0497	-0.0085
17	3	0.0128	0.0534	-0.0091
17	4	0.0128	0.0580	-0.0097
17	5	0.0129	0.0647	-0.0102
17	6	0.0130	0.0716	-0.0111
17	7	0.0131	0.0795	-0.0120
17	8	0.0132	0.0914	-0.0133
18	0	0.0105	0.0421	-0.0067
18	1	0.0106	0.0444	-0.0070
18	2	0.0106	0.0470	-0.0074
18	3	0.0107	0.0503	-0.0077
18	4	0.0107	0.0541	-0.0081
18	5	0.0108	0.0594	-0.0087
18	6	0.0108	0.0661	-0.0094
18	7	0.0110	0.0738	-0.0104
18	8	0.0110	0.0831	-0.0113
18	9	0.0112	0.0931	-0.0125
19	0	0.0099	0.0400	-0.0060
19	1	0.0099	0.0416	-0.0062
19	2	0.0100	0.0439	-0.0064
19	3	0.0100	0.0468	-0.0066
19	4	0.0100	0.0500	-0.0069
19	5	0.0101	0.0541	-0.0074
19	6	0.0101	0.0602	-0.0081
19	7	0.0102	0.0660	-0.0085
19	8	0.0103	0.0733	-0.0096
19	9	0.0104	0.0823	-0.0104
20	0	0.0089	0.0377	-0.0055
20	1	0.0090	0.0394	-0.0058
20	2	0.0090	0.0417	-0.0060
20	3	0.0090	0.0445	-0.0062
20	4	0.0090	0.0473	-0.0065
20	5	0.0091	0.0517	-0.0069
20	6	0.0091	0.0559	-0.0073
20	7	0.0092	0.0612	-0.0078
20	8	0.0092	0.0670	-0.0084
20	9	0.0093	0.0729	-0.0091
20	10	0.0094	0.0815	-0.0098
21	0	0.0081	0.0353	-0.0049
21	1	0.0082	0.0368	-0.0051
21	2	0.0082	0.0387	-0.0053
21	3	0.0082	0.0410	-0.0055
21	4	0.0082	0.0432	-0.0057
21	5	0.0082	0.0462	-0.0060
21	6	0.0083	0.0499	-0.0064
21	7	0.0083	0.0544	-0.0067
21	8	0.0083	0.0599	-0.0071
21	9	0.0084	0.0660	-0.0077
21	10	0.0084	0.0731	-0.0083
22	0	0.0075	0.0347	-0.0045
22	1	0.0075	0.0361	-0.0046
22	2	0.0075	0.0378	-0.0048
22	3	0.0075	0.0398	-0.0050
22	4	0.0076	0.0418	-0.0053
22	5	0.0076	0.0448	-0.0056
22	6	0.0076	0.0481	-0.0059
22	7	0.0076	0.0517	-0.0061
22	8	0.0077	0.0562	-0.0066
22	9	0.0077	0.0623	-0.0071
22	10	0.0078	0.0675	-0.0076
22	11	0.0078	0.0755	-0.0084
23	0	0.0071	0.0331	-0.0044
23	1	0.0071	0.0343	-0.0046
23	2	0.0071	0.0360	-0.0048
23	3	0.0071	0.0379	-0.0050
23	4	0.0071	0.0398	-0.0051
23	5	0.0071	0.0418	-0.0053
23	6	0.0071	0.0450	-0.0054
23	7	0.0072	0.0479	-0.0057
23	8	0.0072	0.0520	-0.0060
23	9	0.0072	0.0567	-0.0064
23	10	0.0073	0.0620	-0.0069
23	11	0.0073	0.0677	-0.0074
24	0	0.0063	0.0305	-0.0038
24	1	0.0064	0.0316	-0.0039
24	2	0.0064	0.0329	-0.0040
24	3	0.0064	0.0348	-0.0042
24	4	0.0064	0.0368	-0.0044
24	5	0.0064	0.0386	-0.0046
24	6	0.0064	0.0412	-0.0048
24	7	0.0065	0.0442	-0.0051
24	8	0.0065	0.0478	-0.0053
24	9	0.0065	0.0515	-0.0057
24	10	0.0066	0.0562	-0.0061
24	11	0.0066	0.0614	-0.0066
24	12	0.0067	0.0679	-0.0072
25	0	0.0059	0.0298	-0.0038
25	1	0.0059	0.0307	-0.0038
25	2	0.0059	0.0321	-0.0040
25	3	0.0059	0.0336	-0.0041
25	4	0.0059	0.0351	-0.0042
25	5	0.0059	0.0371	-0.0044
25	6	0.0059	0.0393	-0.0045
25	7	0.0060	0.0418	-0.0047
25	8	0.0060	0.0450	-0.0050
25	9	0.0060	0.0488	-0.0053
25	10	0.0060	0.0523	-0.0056
25	11	0.0061	0.0564	-0.0059
25	12	0.0061	0.0611	-0.0063
26	0	0.0054	0.0279	-0.0033
26	1	0.0054	0.0287	-0.0033
26	2	0.0054	0.0298	-0.0035
26	3	0.0054	0.0310	-0.0036
26	4	0.0055	0.0323	-0.0036
26	5	0.0055	0.0341	-0.0037
26	6	0.0055	0.0360	-0.0039
26	7	0.0055	0.0385	-0.0041
26	8	0.0055	0.0410	-0.0043
26	9	0.0055	0.0440	-0.0045
26	10	0.0055	0.0478	-0.0048
26	11	0.0056	0.0508	-0.0051
26	12	0.0056	0.0555	-0.0053
26	13	0.0056	0.0613	-0.0059
27	0	0.0053	0.0271	-0.0031
27	1	0.0053	0.0277	-0.0032
27	2	0.0053	0.0288	-0.0034
27	3	0.0053	0.0299	-0.0034
27	4	0.0053	0.0314	-0.0035
27	5	0.0053	0.0326	-0.0036
27	6	0.0053	0.0343	-0.0037
27	7	0.0053	0.0360	-0.0038
27	8	0.0053	0.0385	-0.0039
27	9	0.0054	0.0412	-0.0042
27	10	0.0054	0.0437	-0.0043
27	11	0.0054	0.0473	-0.0046
27	12	0.0054	0.0506	-0.0048
27	13	0.0054	0.0551	-0.0052
28	0	0.0048	0.0264	-0.0031
28	1	0.0048	0.0271	-0.0031
28	2	0.0048	0.0282	-0.0033
28	3	0.0048	0.0293	-0.0033
28	4	0.0048	0.0307	-0.0034
28	5	0.0049	0.0318	-0.0036
28	6	0.0049	0.0333	-0.0036
28	7	0.0049	0.0351	-0.0038
28	8	0.0049	0.0373	-0.0038
28	9	0.0049	0.0400	-0.0041
28	10	0.0049	0.0434	-0.0043
28	11	0.0049	0.0467	-0.0046
28	12	0.0049	0.0496	-0.0048
28	13	0.0050	0.0535	-0.0051
28	14	0.0050	0.0579	-0.0054
29	0	0.0046	0.0255	-0.0028
29	1	0.0046	0.0262	-0.0029
29	2	0.0046	0.0273	-0.0030
29	3	0.0046	0.0283	-0.0031
29	4	0.0046	0.0296	-0.0032
29	5	0.0046	0.0306	-0.0032
29	6	0.0046	0.0318	-0.0033
29	7	0.0046	0.0337	-0.0035
29	8	0.0046	0.0358	-0.0036
29	9	0.0047	0.0382	-0.0038
29	10	0.0047	0.0408	-0.0040
29	11	0.0047	0.0433	-0.0041
29	12	0.0047	0.0461	-0.0043
29	13	0.0047	0.0491	-0.0046
29	14	0.0048	0.0533	-0.0050
30	0	0.0044	0.0246	-0.0025
30	1	0.0044	0.0253	-0.0026
30	2	0.0044	0.0261	-0.0027
30	3	0.0044	0.0271	-0.0027
30	4	0.0044	0.0283	-0.0028

Table 4 (continued)

n	s	$\frac{V(\mu^*)}{\sigma^2}$	$\frac{V(\sigma^*)}{\sigma^2}$	$\frac{Cov(\mu^*,\sigma^*)}{\sigma^2}$
30	5	0.0044	0.0292	-0.0029
30	6	0.0044	0.0302	-0.0029
30	7	0.0044	0.0318	-0.0030
30	8	0.0044	0.0337	-0.0031
30	9	0.0044	0.0357	-0.0032
30	10	0.0044	0.0377	-0.0033
30	11	0.0044	0.0405	-0.0035
30	12	0.0044	0.0437	-0.0037
30	13	0.0045	0.0463	-0.0039
30	14	0.0045	0.0495	-0.0042
30	15	0.0045	0.0534	-0.0045
31	0	0.0040	0.0238	-0.0023
31	1	0.0040	0.0243	-0.0023
31	2	0.0040	0.0250	-0.0024
31	3	0.0040	0.0260	-0.0025
31	4	0.0040	0.0271	-0.0025
31	5	0.0040	0.0282	-0.0026
31	6	0.0040	0.0291	-0.0026
31	7	0.0040	0.0306	-0.0027
31	8	0.0040	0.0321	-0.0028
31	9	0.0040	0.0339	-0.0029
31	10	0.0040	0.0362	-0.0031
31	11	0.0041	0.0387	-0.0033
31	12	0.0041	0.0406	-0.0034
31	13	0.0041	0.0430	-0.0035
31	14	0.0041	0.0452	-0.0038
31	15	0.0041	0.0485	-0.0039
32	0	0.0039	0.0227	-0.0023
32	1	0.0039	0.0232	-0.0023
32	2	0.0039	0.0241	-0.0024
32	3	0.0039	0.0250	-0.0025
32	4	0.0039	0.0257	-0.0025
32	5	0.0039	0.0266	-0.0026
32	6	0.0039	0.0278	-0.0026
32	7	0.0039	0.0290	-0.0028
32	8	0.0039	0.0305	-0.0028
32	9	0.0040	0.0318	-0.0029
32	10	0.0040	0.0335	-0.0031
32	11	0.0040	0.0356	-0.0032
32	12	0.0040	0.0378	-0.0033
32	13	0.0040	0.0401	-0.0035
32	14	0.0040	0.0423	-0.0037
32	15	0.0040	0.0454	-0.0039
32	16	0.0040	0.0485	-0.0041
33	0	0.0037	0.0222	-0.0020
33	1	0.0037	0.0227	-0.0021
33	2	0.0037	0.0235	-0.0021
33	3	0.0037	0.0243	-0.0022
33	4	0.0037	0.0253	-0.0022
33	5	0.0037	0.0261	-0.0023
33	6	0.0037	0.0268	-0.0024
33	7	0.0037	0.0280	-0.0025
33	8	0.0037	0.0293	-0.0026
33	9	0.0037	0.0308	-0.0026
33	10	0.0037	0.0326	-0.0027
33	11	0.0037	0.0342	-0.0029
33	12	0.0037	0.0363	-0.0030
33	13	0.0037	0.0383	-0.0031
33	14	0.0037	0.0407	-0.0032
33	15	0.0038	0.0433	-0.0034
33	16	0.0038	0.0470	-0.0037
34	0	0.0034	0.0212	-0.0020
34	1	0.0034	0.0217	-0.0020
34	2	0.0034	0.0225	-0.0021
34	3	0.0034	0.0231	-0.0021
34	4	0.0034	0.0239	-0.0021
34	5	0.0034	0.0247	-0.0022
34	6	0.0034	0.0255	-0.0023
34	7	0.0034	0.0265	-0.0023
34	8	0.0034	0.0277	-0.0023
34	9	0.0035	0.0290	-0.0024
34	10	0.0035	0.0306	-0.0025
34	11	0.0035	0.0321	-0.0026
34	12	0.0035	0.0337	-0.0027
34	13	0.0035	0.0357	-0.0029
34	14	0.0035	0.0375	-0.0030
34	15	0.0035	0.0399	-0.0031
34	16	0.0035	0.0419	-0.0032
34	17	0.0035	0.0451	-0.0034
35	0	0.0032	0.0205	-0.0018
35	1	0.0032	0.0210	-0.0018
35	2	0.0032	0.0216	-0.0019
35	3	0.0032	0.0222	-0.0019
35	4	0.0032	0.0229	-0.0020
35	5	0.0032	0.0237	-0.0020
35	6	0.0032	0.0246	-0.0021
35	7	0.0032	0.0254	-0.0021
35	8	0.0032	0.0262	-0.0021
35	9	0.0032	0.0278	-0.0022
35	10	0.0032	0.0291	-0.0023
35	11	0.0032	0.0306	-0.0024
35	12	0.0033	0.0325	-0.0025
35	13	0.0033	0.0342	-0.0026
35	14	0.0033	0.0360	-0.0027
35	15	0.0033	0.0379	-0.0028
35	16	0.0033	0.0405	-0.0030
35	17	0.0033	0.0425	-0.0031
36	0	0.0031	0.0199	-0.0019
36	1	0.0031	0.0205	-0.0020
36	2	0.0031	0.0210	-0.0020
36	3	0.0031	0.0217	-0.0020
36	4	0.0031	0.0225	-0.0021
36	5	0.0031	0.0233	-0.0021
36	6	0.0031	0.0238	-0.0022
36	7	0.0031	0.0247	-0.0022
36	8	0.0031	0.0256	-0.0022
36	9	0.0031	0.0267	-0.0023
36	10	0.0031	0.0279	-0.0023
36	11	0.0031	0.0294	-0.0024
36	12	0.0031	0.0307	-0.0025
36	13	0.0031	0.0322	-0.0026
36	14	0.0031	0.0341	-0.0027
36	15	0.0031	0.0359	-0.0028
36	16	0.0031	0.0376	-0.0029
36	17	0.0032	0.0398	-0.0030
36	18	0.0032	0.0423	-0.0032
37	0	0.0030	0.0194	-0.0018
37	1	0.0030	0.0199	-0.0018
37	2	0.0030	0.0204	-0.0018
37	3	0.0030	0.0210	-0.0018
37	4	0.0031	0.0217	-0.0019
37	5	0.0031	0.0223	-0.0019
37	6	0.0031	0.0231	-0.0020
37	7	0.0031	0.0237	-0.0020
37	8	0.0031	0.0244	-0.0020
37	9	0.0031	0.0253	-0.0021
37	10	0.0031	0.0266	-0.0021
37	11	0.0031	0.0280	-0.0022
37	12	0.0031	0.0294	-0.0023
37	13	0.0031	0.0308	-0.0024
37	14	0.0031	0.0328	-0.0025
37	15	0.0031	0.0343	-0.0026
37	16	0.0031	0.0361	-0.0027
37	17	0.0031	0.0379	-0.0029
37	18	0.0031	0.0403	-0.0031
38	0	0.0027	0.0188	-0.0016
38	1	0.0027	0.0193	-0.0016
38	2	0.0027	0.0198	-0.0017
38	3	0.0027	0.0204	-0.0017
38	4	0.0027	0.0211	-0.0017
38	5	0.0027	0.0217	-0.0018
38	6	0.0027	0.0225	-0.0018
38	7	0.0027	0.0232	-0.0019
38	8	0.0027	0.0240	-0.0019
38	9	0.0027	0.0247	-0.0019
38	10	0.0027	0.0257	-0.0020
38	11	0.0027	0.0269	-0.0020
38	12	0.0027	0.0282	-0.0021
38	13	0.0028	0.0298	-0.0022
38	14	0.0028	0.0315	-0.0023
38	15	0.0028	0.0330	-0.0024
38	16	0.0028	0.0349	-0.0024
38	17	0.0028	0.0368	-0.0026
38	18	0.0028	0.0388	-0.0027
38	19	0.0028	0.0411	-0.0029
39	0	0.0025	0.0184	-0.0016
39	1	0.0025	0.0189	-0.0016
39	2	0.0025	0.0195	-0.0016
39	3	0.0025	0.0200	-0.0017
39	4	0.0025	0.0206	-0.0017
39	5	0.0025	0.0213	-0.0017
39	6	0.0025	0.0220	-0.0018
39	7	0.0025	0.0226	-0.0018
39	8	0.0025	0.0235	-0.0018
39	9	0.0025	0.0243	-0.0019
39	10	0.0025	0.0252	-0.0019
39	11	0.0025	0.0262	-0.0020
39	12	0.0025	0.0274	-0.0020
39	13	0.0025	0.0287	-0.0021
39	14	0.0025	0.0304	-0.0022
39	15	0.0025	0.0321	-0.0023
39	16	0.0025	0.0336	-0.0024
39	17	0.0025	0.0351	-0.0024
39	18	0.0025	0.0370	-0.0025
39	19	0.0025	0.0390	-0.0027
40	0	0.0024	0.0178	-0.0014
40	1	0.0024	0.0182	-0.0015
40	2	0.0024	0.0188	-0.0015
40	3	0.0024	0.0193	-0.0015
40	4	0.0024	0.0199	-0.0016
40	5	0.0024	0.0204	-0.0016
40	6	0.0024	0.0210	-0.0016
40	7	0.0024	0.0215	-0.0016
40	8	0.0024	0.0222	-0.0017
40	9	0.0024	0.0229	-0.0017
40	10	0.0024	0.0239	-0.0018
40	11	0.0024	0.0248	-0.0018
40	12	0.0025	0.0260	-0.0019
40	13	0.0025	0.0274	-0.0020
40	14	0.0025	0.0287	-0.0021
40	15	0.0025	0.0298	-0.0022
40	16	0.0025	0.0314	-0.0023
40	17	0.0025	0.0331	-0.0023
40	18	0.0025	0.0345	-0.0024
40	19	0.0025	0.0369	-0.0025
40	20	0.0025	0.0389	-0.0026
41	0	0.0023	0.0178	-0.0014
41	1	0.0023	0.0182	-0.0014
41	2	0.0023	0.0187	-0.0014
41	3	0.0023	0.0191	-0.0014
41	4	0.0023	0.0196	-0.0014
41	5	0.0023	0.0202	-0.0014
41	6	0.0023	0.0208	-0.0015
41	7	0.0023	0.0213	-0.0015
41	8	0.0023	0.0219	-0.0015
41	9	0.0023	0.0226	-0.0016
41	10	0.0023	0.0233	-0.0016
41	11	0.0023	0.0243	-0.0016
41	12	0.0023	0.0254	-0.0017
41	13	0.0023	0.0265	-0.0017
41	14	0.0023	0.0278	-0.0019
41	15	0.0023	0.0291	-0.0019
41	16	0.0023	0.0307	-0.0020
41	17	0.0023	0.0322	-0.0021
41	18	0.0023	0.0338	-0.0021
41	19	0.0024	0.0356	-0.0022
41	20	0.0024	0.0372	-0.0023

Table 4 (continued)

n	s	$\frac{V(\mu^*)}{\sigma^2}$	$\frac{V(\sigma^*)}{\sigma^2}$	$\frac{Cov(\mu^*,\sigma^*)}{\sigma^2}$
42	0	0.0022	0.0167	-0.0013
42	1	0.0022	0.0171	-0.0013
42	2	0.0022	0.0174	-0.0013
42	3	0.0022	0.0179	-0.0014
42	4	0.0022	0.0185	-0.0014
42	5	0.0022	0.0189	-0.0014
42	6	0.0022	0.0195	-0.0014
42	7	0.0022	0.0199	-0.0015
42	8	0.0022	0.0206	-0.0015
42	9	0.0022	0.0212	-0.0015
42	10	0.0022	0.0219	-0.0015
42	11	0.0022	0.0228	-0.0016
42	12	0.0022	0.0237	-0.0016
42	13	0.0022	0.0249	-0.0017
42	14	0.0022	0.0260	-0.0018
42	15	0.0022	0.0273	-0.0018
42	16	0.0022	0.0285	-0.0019
42	17	0.0022	0.0300	-0.0019
42	18	0.0022	0.0313	-0.0020
42	19	0.0022	0.0328	-0.0021
42	20	0.0022	0.0346	-0.0022
42	21	0.0022	0.0364	-0.0023
43	0	0.0021	0.0165	-0.0013
43	1	0.0021	0.0169	-0.0013
43	2	0.0021	0.0173	-0.0013
43	3	0.0021	0.0177	-0.0013
43	4	0.0021	0.0182	-0.0014
43	5	0.0021	0.0186	-0.0014
43	6	0.0021	0.0192	-0.0014
43	7	0.0021	0.0197	-0.0014
43	8	0.0021	0.0202	-0.0014
43	9	0.0021	0.0207	-0.0015
43	10	0.0021	0.0214	-0.0015
43	11	0.0021	0.0221	-0.0015
43	12	0.0021	0.0229	-0.0015
43	13	0.0021	0.0241	-0.0016
43	14	0.0021	0.0251	-0.0016
43	15	0.0021	0.0262	-0.0017
43	16	0.0021	0.0275	-0.0018
43	17	0.0021	0.0288	-0.0018
43	18	0.0021	0.0301	-0.0019
43	19	0.0021	0.0314	-0.0019
43	20	0.0021	0.0336	-0.0020
43	21	0.0021	0.0354	-0.0022
44	0	0.0020	0.0163	-0.0012
44	1	0.0020	0.0166	-0.0012
44	2	0.0020	0.0170	-0.0012
44	3	0.0020	0.0174	-0.0013
44	4	0.0020	0.0178	-0.0013
44	5	0.0020	0.0183	-0.0013
44	6	0.0020	0.0188	-0.0013
44	7	0.0020	0.0193	-0.0013
44	8	0.0020	0.0198	-0.0013
44	9	0.0020	0.0203	-0.0014
44	10	0.0020	0.0209	-0.0014
44	11	0.0020	0.0217	-0.0014
44	12	0.0020	0.0226	-0.0015
44	13	0.0020	0.0234	-0.0015
44	14	0.0020	0.0244	-0.0015
44	15	0.0020	0.0254	-0.0016
44	16	0.0020	0.0266	-0.0017
44	17	0.0020	0.0278	-0.0017
44	18	0.0020	0.0292	-0.0018
44	19	0.0020	0.0302	-0.0018
44	20	0.0020	0.0315	-0.0019
44	21	0.0020	0.0331	-0.0020
44	22	0.0020	0.0347	-0.0021
45	0	0.0019	0.0158	-0.0012
45	1	0.0019	0.0161	-0.0012
45	2	0.0019	0.0164	-0.0012
45	3	0.0019	0.0168	-0.0013
45	4	0.0019	0.0172	-0.0013
45	5	0.0019	0.0176	-0.0013
45	6	0.0019	0.0179	-0.0013
45	7	0.0019	0.0184	-0.0013
45	8	0.0019	0.0189	-0.0014
45	9	0.0019	0.0194	-0.0014
45	10	0.0020	0.0201	-0.0014
45	11	0.0020	0.0207	-0.0015
45	12	0.0020	0.0213	-0.0015
45	13	0.0020	0.0222	-0.0015
45	14	0.0020	0.0233	-0.0016
45	15	0.0020	0.0243	-0.0017
45	16	0.0020	0.0253	-0.0017
45	17	0.0020	0.0263	-0.0017
45	18	0.0020	0.0275	-0.0017
45	19	0.0020	0.0288	-0.0018
45	20	0.0020	0.0302	-0.0019
45	21	0.0020	0.0317	-0.0020
45	22	0.0020	0.0336	-0.0021
46	0	0.0019	0.0153	-0.0012
46	1	0.0019	0.0157	-0.0012
46	2	0.0019	0.0161	-0.0012
46	3	0.0019	0.0164	-0.0012
46	4	0.0019	0.0168	-0.0012
46	5	0.0019	0.0171	-0.0012
46	6	0.0019	0.0175	-0.0013
46	7	0.0019	0.0179	-0.0013
46	8	0.0019	0.0183	-0.0013
46	9	0.0019	0.0189	-0.0013
46	10	0.0019	0.0195	-0.0013
46	11	0.0019	0.0201	-0.0014
46	12	0.0019	0.0209	-0.0014
46	13	0.0019	0.0218	-0.0015
46	14	0.0019	0.0226	-0.0015
46	15	0.0019	0.0234	-0.0015
46	16	0.0019	0.0245	-0.0016
46	17	0.0019	0.0257	-0.0016
46	18	0.0019	0.0267	-0.0017
46	19	0.0019	0.0278	-0.0017
46	20	0.0019	0.0290	-0.0018
46	21	0.0019	0.0303	-0.0018
46	22	0.0019	0.0320	-0.0019
46	23	0.0019	0.0335	-0.0020
47	0	0.0018	0.0149	-0.0011
47	1	0.0018	0.0152	-0.0011
47	2	0.0018	0.0156	-0.0011
47	3	0.0018	0.0160	-0.0011
47	4	0.0018	0.0163	-0.0011
47	5	0.0018	0.0167	-0.0011
47	6	0.0018	0.0172	-0.0012
47	7	0.0018	0.0176	-0.0012
47	8	0.0018	0.0180	-0.0012
47	9	0.0018	0.0184	-0.0012
47	10	0.0018	0.0189	-0.0012
47	11	0.0018	0.0194	-0.0013
47	12	0.0018	0.0199	-0.0012
47	13	0.0018	0.0207	-0.0013
47	14	0.0018	0.0215	-0.0014
47	15	0.0018	0.0224	-0.0014
47	16	0.0018	0.0233	-0.0015
47	17	0.0018	0.0243	-0.0015
47	18	0.0018	0.0255	-0.0015
47	19	0.0018	0.0264	-0.0015
47	20	0.0018	0.0273	-0.0016
47	21	0.0018	0.0286	-0.0016
47	22	0.0018	0.0300	-0.0017
47	23	0.0018	0.0316	-0.0018
48	0	0.0018	0.0151	-0.0011
48	1	0.0018	0.0154	-0.0011
48	2	0.0018	0.0158	-0.0011
48	3	0.0018	0.0160	-0.0011
48	4	0.0018	0.0164	-0.0011
48	5	0.0018	0.0167	-0.0011
48	6	0.0018	0.0171	-0.0011
48	7	0.0018	0.0176	-0.0011
48	8	0.0018	0.0179	-0.0011
48	9	0.0018	0.0183	-0.0011
48	10	0.0018	0.0188	-0.0012
48	11	0.0018	0.0194	-0.0012
48	12	0.0018	0.0199	-0.0012
48	13	0.0018	0.0205	-0.0012
48	14	0.0018	0.0211	-0.0013
48	15	0.0018	0.0220	-0.0013
48	16	0.0018	0.0230	-0.0013
48	17	0.0018	0.0239	-0.0014
48	18	0.0018	0.0248	-0.0014
48	19	0.0018	0.0258	-0.0015
48	20	0.0018	0.0268	-0.0015
48	21	0.0018	0.0278	-0.0015
48	22	0.0018	0.0292	-0.0016
48	23	0.0018	0.0304	-0.0016
48	24	0.0018	0.0320	-0.0017
49	0	0.0016	0.0144	-0.0010
49	1	0.0016	0.0147	-0.0010
49	2	0.0016	0.0150	-0.0010
49	3	0.0016	0.0154	-0.0010
49	4	0.0016	0.0157	-0.0011
49	5	0.0016	0.0160	-0.0011
49	6	0.0016	0.0164	-0.0011
49	7	0.0016	0.0168	-0.0011
49	8	0.0016	0.0172	-0.0011
49	9	0.0016	0.0176	-0.0011
49	10	0.0017	0.0181	-0.0011
49	11	0.0017	0.0187	-0.0011
49	12	0.0017	0.0192	-0.0012
49	13	0.0017	0.0197	-0.0012
49	14	0.0017	0.0204	-0.0012
49	15	0.0017	0.0212	-0.0013
49	16	0.0017	0.0221	-0.0013
49	17	0.0017	0.0230	-0.0014
49	18	0.0017	0.0239	-0.0014
49	19	0.0017	0.0249	-0.0014
49	20	0.0017	0.0259	-0.0015
49	21	0.0017	0.0270	-0.0015
49	22	0.0017	0.0279	-0.0016
49	23	0.0017	0.0293	-0.0016
49	24	0.0017	0.0307	-0.0017
50	0	0.0016	0.0143	-0.0010
50	1	0.0016	0.0146	-0.0010
50	2	0.0016	0.0150	-0.0010
50	3	0.0016	0.0153	-0.0010
50	4	0.0016	0.0156	-0.0011
50	5	0.0016	0.0160	-0.0011
50	6	0.0016	0.0163	-0.0011
50	7	0.0016	0.0167	-0.0011
50	8	0.0016	0.0171	-0.0011
50	9	0.0016	0.0174	-0.0011
50	10	0.0016	0.0180	-0.0011
50	11	0.0016	0.0184	-0.0012
50	12	0.0016	0.0190	-0.0012
50	13	0.0016	0.0196	-0.0012
50	14	0.0016	0.0203	-0.0012
50	15	0.0016	0.0211	-0.0013
50	16	0.0016	0.0218	-0.0013
50	17	0.0016	0.0226	-0.0013
50	18	0.0016	0.0232	-0.0013
50	19	0.0016	0.0241	-0.0014
50	20	0.0016	0.0249	-0.0014
50	21	0.0016	0.0259	-0.0014
50	22	0.0016	0.0270	-0.0015
50	23	0.0016	0.0282	-0.0015
50	24	0.0016	0.0296	-0.0016
50	25	0.0016	0.0311	-0.0017

ILLUSTRATIVE EXAMPLES

Example 1(Real data): Consider the following Type-II censored data which represent failure times, in minutes, for a specific type of electrical insulation in an experiment in which the insulation was subjected to a continuously increasing voltage stress (Lawless[12], p.138):

12.3, 21.8, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6.

Here, the largest observation is censored sine the experiment was stopped at 11th failure. Balakrishnan and Puthenpura[6] have analyzed this data and concluded that the half-logistic distribution fits the data better than an exponential distribution. Later, Balakrishnan and Wong[7] used the above data to demonstrate the estimation of AMLE of σ and unbiased AMLEs of μ and σ and mean failure time ($\mu + \sigma \log_e 4$). For comparison purpose, we compute the estimates MLE, AMLE and LAMLE of σ; and presented them in **Table I** along with the standard errors (S.E.s) of AMLE and LAMLE for all possible Type-II right censored samples constructed from the given sample.

In the **Table I**, the S.E. of AMLE is computed based on the simulated variance of AMLE ($Var(\breve{\sigma})/\sigma^2$) given in 'Table 3' of Balakrishnan and Wong[7], whereas the S.E. of LAMLE is based on the exact variance of the estimator using Eq. [14]. From the above table, we may notice that LAMLE is slightly smaller than MLE where as AMLE is slightly larger than MLE which is perhaps due to the fact that, as is observed from ***Table 2***, AMLE is less biased and LAMLE is more biased than MLE. But, at the same time, we may notice that the LAMLE has slightly smaller S.E. as compared with AMLE. Thus, this example ascertain our empirical conclusion that AMLE is slightly less biased and less efficient, where as LAMLE is slightly more biased and more efficient than MLE. Similarly, we compute the BLUEs, unbiased AMLEs and unbiased LAMLEs of μ, σ and mean failure time; and are presented in **Table II** along with the S.E.s of the estimators of mean failure time for all possible Type-II right censored samples constructed from the given sample.

Table I: Comparison of AMLE and LAMLEs of σof half-logistic distribution with the corresponding MLE computed from the given sample of size n=12, for s=1(1)n-2

S	MLE	AMLE	(S.E.)	LAMLE	(S.E.)	S	MLE	AMLE	(S.E.)	LAMLE	(S.E.)
1	42.45	42.66	(10.54)	42.30	(10.18)	6	32.19	32.22	(10.29)	32.08	(10.13)
2	39.69	39.83	(10.23)	39.52	(9.89)	7	34.67	34.70	(12.07)	34.51	(11.82)
3	42.74	42.86	(11.42)	42.52	(11.14)	8	24.12	24.14	(9.13)	24.03	(9.03)
4	39.84	39.92	(11.22)	39.63	(10.96)	9	24.18	24.20	(10.09)	24.05	(9.98)
5	42.52	42.61	(12.71)	42.20	(12.42)	10	28.42	28.46	(12.92)	28.26	(12.63)

Table II: Comparison of unbiased AMLEs and unbiased LAMLEs of location (μ), scale (σ) and mean failure time of half-logistic distribution with the corresponding BLUEs computed from the give sample of size n=12, for s=1(1)n-2.

S	LOCATION PARAMETER			SCALE PARAMETER			MEAN FAILURE TIME		
	BLUE	UAMLE	ULAMLE	BLUE	UAMLE	ULAMLE	BLUE (S.E.)	UAMLE (S.E.)	ULAMLE (S.E.)
1	4.84	5.00	4.93	47.44	46.97	47.43	70.61(16.53)	70.11(16.36)	70.69(16.54)
2	5.34	5.39	5.33	44.88	44.49	44.89	67.56(16.35)	67.06(16.27)	67.55(16.36)
3	4.56	4.74	4.68	49.07	48.65	49.04	72.59(18.87)	72.18(18.45)	72.67(18.87)
4	4.99	5.12	5.06	46.68	46.23	46.58	69.70(19.17)	69.21(18.79)	69.64(19.14)
5	4.26	4.43	4.41	50.88	50.66	50.81	74.79(22.64)	74.66(22.33)	74.84(22.61)
6	6.12	6.14	6.10	39.88	39.63	39.88	61.41(19.56)	61.08(19.26)	61.39(19.56)
7	5.28	5.34	5.33	44.94	44.80	44.88	67.58(24.88)	67.45(24.78)	67.55(24.86)
8	7.12	7.08	7.09	33.53	33.62	33.50	53.61(21.74)	53.69(21.71)	53.54(21.72)
9	6.42	6.36	6.40	37.99	38.23	37.96	59.08(30.70)	59.36(30.92)	59.02(30.67)
10	2.99	2.87	2.99	59.89	60.71	59.89	86.02(69.88)	87.03(71.67)	86.02(69.88)

From the above table, we may notice that when compared with Balakrishnan and Wong's unbiased AMLEs, unbiased LAMLEs of the respective parameters are very much closer to the corresponding BLUEs even in heavy censored samples. We may notice the same observation in respect of the S.E.s of the estimates of mean failure time. Thus, this real example illustrates the theoretical conclusion namely '*the unbiased LAMLEs are almost as efficient as the BLUEs*'.

Example 2: We consider the following example of sample size 50 (simulated from the half-logistic distribution with μ=0 and σ=25), which was generated by Balakrishnan and Asgharzadeh[4].

1.7110	2.0024	2.3963	3.9034	4.6412	6.4002	6.7956	8.5646
8.6428	8.8354	9.3518	9.7358	10.5080	10.5095	11.8015	12.8005
16.3451	16.9938	17.2101	18.5384	20.3508	21.1838	22.1529	22.4062
22.4381	23.0369	25.8435	27.0574	27.1237	29.0360	30.6449	32.5713
33.6688	40.3890	45.4092	46.4756	49.8833	51.1798	53.0397	53.8135
64.9315	66.1807	69.9004	75.2674	75.4427	75.7291	76.1571	89.5827
99.8525	134.6488						

The estimates for the above sample can be computed just as in example 1. For comparison purpose, in **Table III**, given below, we present the estimates MLE, AMLE and LAMLE for complete and Type-II right censored samples computed from the above given sample. We also compute S.E. of LAMLE using Eq. [16] and presented in the table. However, since Balakrishnan and Wong[7] have not tabulated the variances beyond the sample size 20, we could not present S.E. of AMLE.

As sample size is greater than 15, as explained in the above section, we cannot compute BLUEs. However, we compute unbiased LAMLEs μ* and σ* using Eq. [18] and presented them in the table along with the S.E.s, which are computed using the simulated variances, those given in ***Table 4***.

Table III: The estimates MLE, AMLE, LAMLE and unbiased LAMLE of scale parameter and unbiased LAMLE of location parameter of half-logistic distribution computed from complete (s=0) and Type II right censored samples with choice of s=1(1)10(5)15 constructed from the above sample of size n=50.

s	SCALE PARAMETER				LOCATION PARAMETER
	MLE	AMLE	LAMLE (S.E.)	Unbiased LAMLE (S.E.)	Unbiased LAMLE (S.E.)
0	23.753	23.837	23.692 (2.793)	24.175 (2.891)	0.753 (0.967)
1	23.584	23.662	23.525 (2.799)	24.053 (2.876)	0.758 (0.962)
2	23.634	23.712	23.573 (2.829)	24.125 (2.885)	0.755 (0.965)
3	23.371	23.455	23.316 (2.824)	23.879 (2.855)	0.765 (0.955)
4	23.747	23.824	23.680 (2.896)	24.269 (2.902)	0.750 (0.971)
5	24.147	24.222	24.067 (2.973)	24.681 (2.951)	0.733 (0.987)
6	24.572	24.654	24.477 (3.056)	25.119 (3.004)	0.716 (1.005)
7	24.358	24.433	24.266 (3.064)	24.920 (2.980)	0.724 (0.997)
8	24.285	24.357	24.192 (3.090)	24.860 (2.973)	0.726 (0.994)
9	24.565	24.644	24.457 (3.162)	25.152 (3.008)	0.715 (1.006)
10	22.925	22.966	22.864 (2.994)	23.531 (2.814)	0.779 (0.941)
15	23.299	23.342	23.212 (3.267)	23.991 (2.869)	0.761 (0.960)

From the above example, we may notice the similar observations those made in the first example, with regard to MLE, AMLE and LAMLE of scale parameter. LAMLE as well as AMLE are very close to MLE even in censored samples irrespective of moderate or heavy censoring. We may also observe that when compared with other estimates, unbiased LAMLE of scale parameter is closer to the true value of the parameter.

CONCLUSION

Through this study, we succeed to provide some new unbiased linear estimators for location and scale parameters of half-logistic distribution namely unbiased LAMLEs, which are just as efficient as the BLUEs. As discussed in the above section, when sample size is beyond 15, it is very difficult to compute the BLUEs and therefore, in this case unbiased LAMLEs may be used as alternative to the BLUEs in complete samples as well as Type II right censored samples.

REFERENCES

1. A. Adatia, Approximate best linear unbiased estimates of the parameters of the half-logistic distribution based on fairly large doubly censored samples. *Computational statistics and Data Analysis*,1997, **24**, 179 – 191.
2. A. Adatia, Estimation of the parameters of the half-logistic distribution using generalized rank set sampling. *Computational statistics and Data Analysis*, 2000,**33**, 179 – 191.
3. N. Balakrishnan, Order statistics from the half-logistic distribution. *J. Statist. Comput. Simul.*, 1985, **20**, 287 – 309.

4. N. Balakrishnan and A. Asgharzadeh, Inference for the Scaled Half-Logistic Distribution Based on Progressively Type-II Censored Samples.*Commun. Statist. – Theor. Math.*,2005, **34**, 73-87.

5. N. Balakrishnan and P.S. Chan, Estimation for the scaled half logistic distribution under Type-II censoring. *Computational statistics and Data Analysis,* 1992,**13**, 123 – 141.

6. N. Balakrishnan, S. Puthenpura, Best linear unbiased estimation of location and scale parameters of the half-logistic distribution. , *J. Statist. Comput. Simul.*, 1986, **25**, 193 – 204.

7. N. Balakrishnan and K.H.T. Wong, Approximate MLEs for the location and scale parameters of the half logistic distribution with Type-II right censoring. *I E EE Trans. On Reliab.*, 1991,**40**, 140–145.

8. N. Balakrishnan and K.H.T. Wong, Best linear unbiased estimation of location and scale parameters of the half-logistic distribution based on Type-II censored samples. *Amer. J. Math. Mgt. Sci.*, 1994, **14**, 53-101.

9. E. G. David, Bias reduction for the maximum likelihood estimators of the parameters in the half-logistic distribution. *Commun. Statist. –Theor. Math.*, 2012, **41**, 212-222.

10. R.R.L. Kantam and K. Rosaiah, Half-logistic distribution in acceptance sampling based on life tests. *IAPQR Transactions,* 1998,**23**, 117 – 125.

11. R.R.L.Kantam, K. Rosaiah and M.S.R. Anjaneyulu, Estimation of Reliability in multi component stress – strength model: half-logistic distribution. *IAPQR Transactions*, 2000,**25**, 43–52.

12. J.F. Lawless, *Statistical Models and Methods for Life TimeData*. John Wiley & Sons, Newyork, . 1982.

13. E.H. Lloyd, Least squares estimation of location and scale parameters using order statistics. *Biometrika,* 1952, **39**, 88- 95.

14. A.K. Mbah and C.P. Tsokos, Record values from half logistics and inverse Weibull probability distribution functions. *Neural, Parallel Science Computation,* 2008, ***16***, 73-92.

15. A.T. Paul and C.T. David *Applied Reliability*, Second Edition. Van No strand Reinhold, New York., 1995.

16. K.Rosaiah, R.R.L. Kantam and B. Jayaprakash, Reliability estimation in half-logistic model. *Gujarat Statistical Review,* 1997, **24,** 11 – 18.

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