FUZZY CONVEX SUB L-RINGS

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Abstract: The aim of this paper is to introduce the notions of L-fuzzy sub *l*-rings, L-fuzzy convex sub *l*-rings, L-fuzzy prime convex sub *l*-rings, L-fuzzy prime convex sub *l*-rings, L-fuzzy *l*-ideals of *l*-ring with values in a complete lattice which is infinite meet distributive and investigate some properties.

Keywords: L-fuzzy sub *l*-ring, -fuzzy convex sub *l*-rings, -fuzzy prime convex sub *l*-rings, L-fuzzy *l*-ideals.

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Introduction: Ever since L.A.Zadeh introduced the notion of fuzzy sets, the theory of fuzzy sets has attracted several researchers in the areas of Mathematics, Computer Science, Engineering and Technology. J.A.Goguen initiated a more abstract study of fuzzy sets by replacing the values set [0,1] by a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Most of the authors considered fuzzy subsets taking values in a complete lattice. Fuzzy algebra is now a well developed part of algebra. Partially ordered algebraic systems play an important role in algebra. Especially *l*-groups, *l*-rings, Vector lattices and f-rings are important concepts in algebra which present an abstract study of rings of continuous functions. In [13], we introduced L-fuzzy sub *l*-groups and L-fuzzy *l*-ideals. In [14] we introduced Fuzzy Convex sub *l*-groups and in , [15] we studied L-fuzzy prime spectrum of *l*-groups. The objective of this paper is to study L-fuzzy convex sub *l*-rings which assume values in a complete lattice which satisfies infinite meet distributive law.

In this paper, we introduce the concepts of L-fuzzy convex sub *l*-rings, -fuzzy prime convex sub *l*-rings and L-fuzzy maximal convex sub *l*-rings, L-fuzzy *l*-ideals of *l*-rings.

Throughout this paper, let $R \neq 0$ be an *l*-ring and L stands for a nontrivial complete lattice in which the infinite meet distributive law, $a \wedge (\vee_{s \in S} s) = \vee_{s \in S}(a \wedge s)$for any $S \subseteq L$ and $a \in L$ holds. Throughout the paper we consider meet irreducible elements of L only.

1. Preliminaries:

Definition 1.1 : A lattice ordered group is a system $R = (R. +, . , \leq)$where

(i) (R,+) is an abelian group,

(ii) $(R, \leq)$ is a lattice.

Definition 1.2: An L-Fuzzy subset λ of X is a mapping from X into L, where L is a complete lattice satisfying the infinite meet distributive law. If L is the unit interval [0,1] of real numbers, there are the usual fuzzy subsets of X. A L-fuzzy subset $\lambda : G \rightarrow L$ is said to be a nonempty, if it is not the constant map which assumes the values 0 of L.

Definition 1.3: Let $\lambda : X \to L$ be a L-fuzzy subset of X. Then the set $\{\lambda(x) \,/\, x \in X\}$ is called the image of λ and is denoted by $\lambda(x)$ or $Im(\lambda)$. The set $\{ x \,/\, x \in X, \lambda(x) > 0 \}$ is called the support of λ and is denotd by $Supp(\lambda)$. The set $X_\lambda = \{x \in X \,/\, \lambda(x) = \lambda(0)\}$. For $t \in L$, $\lambda_t = \{x \in X \,/\, \lambda(x) \geq t\}$ is called a t-cut or t-level set of λ.

Definition 1.4: Let λ, μ be two L-fuzzy subsets of X. If $\lambda(x) \leq \mu(x)$ for all $x \in X$, then we say that λ is contained in μ and we write $\lambda \subseteq \mu$. Define $\lambda \cup \mu$ and $\lambda \cap \mu$ are L-fuzzy subsets of X by for all $x \in X$, $(\lambda \cup \mu)(x)= \lambda(x) \vee \mu(x)$, $(\lambda \cap \mu) = \lambda(x) \wedge \mu(x)$. Then $\lambda \cup \mu$ and $\lambda \cap \mu$ are called the union and intersection of λ and μ, respectively.

Definition 1.5: Let f be a mapping from X into Y, and let λ and μ be L-fuzzy subsets of X and Y respectively. The L-fuzzy subsets $f(\lambda)$ of Y and $f^{-1}(\mu)$ of X, defined by
$$f(\lambda)(y) = \begin{cases} \vee \{\lambda(x) / \, x \in X, f(x) = y\} \text{ if } f^{-1}(y) \neq \emptyset \\ 0 \qquad otherwise \end{cases}$$
Where $y \in Y$, and $f^{-1}(\mu)(x) = \mu(f(x))$, for all $x \in X$, are called the image of λ under f and the pre-image of μ under f, respectively.

Definition 1.6: A L-fuzzy subset λ of X is said to have sup property if, for any subset A of X, there exists $a_o \in A$ such that $\lambda(a_0) = \vee_{a\in A}\lambda(a)$.

Definition 1.7: Let f be any function from a set X to a set Y, and let λ be any L-fuzzy subset of X. Then λ is called f-invariant if $f(x)=f(y)$ implies $\lambda(x)=\lambda(y)$, where $x,y\in X$.

Definition 1.8: Let X be nonempty set. Let $Y \subseteq X$ and $a \in Y$. We define, a L-fuzzy set a_Y is defined as follows:
$$a_Y(x) = \begin{cases} a & if\ x \in Y \\ 0 & \text{if } x \in X - Y \end{cases}$$
In particular, if Y is a singleton, say, $\{y\}$, then a_y is called as L-fuzzy point. Let R = (R, +, .) be a ring with 0 as the additive identity in R.

Definition 1.9: A L-fuzzy subset λ of R is said to be a L-fuzzy subring of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$, for all $x,y \in R$.

Definition 1.10 : A L-fuzzy subset λ of R is said to be a L-fuzzy ideal of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$, for all $x,y \in R$.

2. L-Fuzzy sub *l*-rings: In this section we introduce the concept of L-fuzzy sub *l*-rings. Here after L stands for a nontrivial complete lattice in which the infinite meet distributive law, $a \wedge (\vee_{s\in S} s) = \vee_{s\in S}(a \wedge s)$ for any $S \subseteq L$ and $a \in L$ holds.
Let R = $(R, +, \vee, \wedge)$ be an *l*-ring with 0 as the additive identity in R.

Definition 2.1: A L-fuzzy subset λ of R is said to be a L-fuzzy sub *l*-ring of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$

iii) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda (y)$

iv) $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda (y)$ for all $x,y\in R$.

Theorem 2.2: Let λ be a fuzzy subset of an *l*-ring R. λ is a L-fuzzy sub *l*-group of R if and only if $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$, $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$ and

$\lambda(x \wedge y) \wedge \lambda(x \vee y) \geq \lambda(x) \wedge \lambda (y)$, for all $x,y \in G$.

Theorem 2.3: A L-fuzzy subset λ of an l-ring R is a L-fuzzy sub l-ring of R if and only if λ_t is a sub l-ring of R for all $t \in \lambda (G) \cup \{t \in L / \lambda(o) \geq t\}$.

Theorem 2.4 : If λ is a L-fuzzy sub l-ring of R, then Supp(λ) is a l-sub ring of R, if Supp (λ) $\neq\phi$ and L is regular. (i.e., if $a \neq o$, $b\neq o \Rightarrow a \wedge b \neq o$ where $a, b \in L$).

Theorem 2.5: If A is any l-sub ring of R, $A \neq G$, then the L-fuzzy subset λ of R defined by

$$\lambda(x) = \begin{cases} s & \text{if } x \in A \\ t & \text{if } x \notin A, \end{cases}$$

where $s, t \in L$ and $t < s \neq o$, is a L-fuzzy sub l-ring of R.

Theorem 2.6: Let λ be a L-fuzzy sub l-ring of an l-ring R. Then
$G_\lambda = \{x \in G/ \lambda(x) = \lambda(0)\}$ is an l-subring of R.

Definition 2.7: Let λ be a L-fuzzy subset of an l-ring of R. Let $\langle\lambda\rangle = \cap\{\mu \mid \lambda \subseteq \mu, \mu$ is any L − fuzzy sub l − ring of R }. Then $\langle\lambda\rangle$ is called the L-fuzzy sub l-ring of R generated by λ. Clearly $\langle\lambda\rangle$ is the smallest L-fuzzy sub l-ring of R which contains λ.

Theorem 2.8: Let R and R^1 be two l-rings. Let λ and μ are two L-fuzzy sub l-rings of R and R^1 respectively. If $f : R \rightarrow R^1$ be a homomorphism and onto then

(i) $f(\lambda)$ is a L-fuzzy sub l-ring of R^1, provided that λ has sup property,
(ii) $f^{-1}(\mu)$ is a L-fuzzy sub l-ring of R,
(iii) $(f(\lambda))(o^1) = \lambda (o)$, where $o^1 \in R^1$ and $o \in R$,
(iv) $f(G_\lambda) \subseteq R^1_{f(\lambda)}$,
(v) If λ is constant on Ker f, then $(f(\lambda))(f(x)) = \lambda (x)$, for all $x \in R$,
(vi) $f^{-1}(R^1_\mu) = R_{f^{-1}(\mu)}$.

As an immediate consequence, if λ is constant on Ker f, it is easy to observe that
i) $f^{-1}(f(\lambda)) = \lambda$ and ii) $f(f^{-1}(\mu)) = \mu$.

3. L-fuzzy convex sub *l*-rings

Definition 3.1: A L-fuzzy sub l-ring λ of R is said to be a L-fuzzy convex sub l-ring of R if $x, a \in G$,
$0 \leq x \leq a \Rightarrow \lambda(x) \geq \lambda(a)$ (Convexity condition)

Theorem 3.2: Let λ be a L-fuzzy sub l-ring of R. Then, λ is a L-fuzzy convex sub l-group of R if and only if $0 \leq x \leq a$ implies $\lambda(0) \geq \lambda(x) \geq \lambda(a)$, for all $x, a \in R$.

Lemma 3.3: Let λ be a L-fuzzy convex sub l-ring. Then, $|x| \leq |a|$ implies
$\lambda(x) \geq \lambda(a)$, for $x, a \in R$.

Theorem 3.4: A L-fuzzy sub l-ring λ of a l-ring R is a L-fuzzy convex sub l-ring of R if and only if for each l-sub ring λ_t, $t \in \lambda(R) \cup \{t \in L \mid \lambda(a) \geq t\}$ is a convex l-sub ring of R. (In fact, for each $t \in L$, λ_t is empty or a convex l-sub ring of R).

Example 3.5: Let $L = [0,1]$. Let $R = Z \times Z$, Where Z be the set of all integers. By ordering lexicographically $(a, b) \geq (0, 0)$ if and only if $a > 0$ *or* $a = 0$ and $b \geq 0$.
Let + be usual addition and $\bullet$ be usual multiplication $(R, +, \bullet, \vee, \wedge)$ is an l-ring with above ordering. Define a L-fuzzy subset $\mu : G \rightarrow L$, by

$$\mu(x) = \begin{cases} 1, & \text{if } (x,y) \in \{(0,0)\} \\ 0.5 & \text{if } (x,y) \in (\{(0,0)\} \times Z) - \{(0,0)\} \\ 0.25, & \text{otherwise} \end{cases}$$

Clearly the level sets $\mu_t = \{(0,0)\}$, if $0.5 < t \leq 1, \mu_t = \{(0,0) \times Z$, if $0.25 < t \leq 0.5$ *and* $\mu_t = G$ for $0 \leq t \leq 0.25, \}$ are convex *l*- sub rings of R. Therefore , μ is a L-fuzzy convex sub *l*-ring of *l*-ring R.

Theorem 3.6: If λ is a L-fuzzy convex sub *l*-ring of R, then $\sup(\lambda) = \{x \in R \mid \lambda(x) > 0\}$ is a convex *l*-sub group of R if $\text{Supp}(\lambda) \neq \emptyset$ and L is regular.

Theorem 3.7 : The intersection of any non empty family of L-fuzzy convex sub *l*-rings of R is a L-fuzzy convex sub *l*-ring.
Theorem3.8. If λ is a L-fuzzy convex sub *l*-ring of R, then $R_\lambda = \{x \in R|\lambda(x) = \lambda(0)\}$ is a convex *l*-subring of R.

Theorem 3.9: If A is any convex sub *l*-subring of R, then the L-fuzzy subset λ of R defined by

$$\lambda(x) = \begin{cases} s \text{ if } x \in A \\ t \text{ if } x \notin A \end{cases}$$

Where s,t ∈L and t < s, is a L-fuzzy convex sub *l*-ring of R.

Theorem 3.10: Let R and R^1 be two *l*-rings. Let λ and μ be L-fuzzy convex sub *l*-rings of R and R^1 respectively. If f: $R \rightarrow R^1$ be a epimorphism, then
(i) f(λ) is a L-fuzzy convex sub *l*-ring of R^1, provided that λ is f- invariant.
(ii) $f^{-1}(\mu)$ is a L-fuzzy convex sub *l*-ring of R.

Theorem.3.11: Let f be a homomorphism of R onto R^1. If λ and μ are L-fuzzy convex sub *l*-rings of R, then $f(\lambda \cap \mu) = f(\lambda) \cap f(\mu)$, provided that if at least one of λ of μ is f – invariant.

Definition3.12: Let λ be a L-fuzzy subset of an *l*-ring R. The smallest L-fuzzy convex sub *l*-ring of R which contains λ is called the L-fuzzy convex sub *l*-ring of R, generated by λ and is denoted by ⟨λ⟩.

Theorem 3.13: Let μ be a L-fuzzy subset of an *l*-ring R. Define ν : R → L be a L-fuzzy subset as follows:

$$\nu(x) = \vee\{\wedge_{y \in A}\mu(y)|A \subseteq R, 1 \leq |A| < \infty, x \in \langle A \rangle\}(x \in R).$$

Where ⟨A⟩ denotes convex *l*-subring generated by A. Then ν = ⟨μ⟩, L-fuzzy convex sub *l*-ring generated by μ.

Definition3.14: Let λ be a L-fuzzy subset of an *l*-ring R. Then λ is called a L-fuzzy maximal convex sub *l*-ring of r, if λ is a maximal element in the set of all non constant L-fuzzy convex sub *l*-rings of R under point wise partial ordering.

Theorem 3.15: Let λ be a L-fuzzy subset of an *l*-ring R. Then λ is a L-fuzzy maximal convex sub *l*-ring of R if and only if there exist, a maximal convex *l*-subring M of R and maximal element α in L such that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in A \\ \alpha, \text{otherwise} \end{cases}$$

Definition 3.16: A non constant L-fuzzy convex sub *l*-ring of an *l*-ring R is called L-fuzzy prime convex sub *l*-ring if and only if for any –fuzzy convex sub *l*-rings μ and ν, $\mu \cap \nu \subseteq \lambda \Rightarrow$ either $\mu \subseteq \lambda$ or $\nu \subseteq \lambda$.

Lemma 3.17: If λ is a L-fuzzy prime convex sub *l*-ring of R, then λ(0) = 1.

Theorem 3.18: Let λ be a L –fuzzy subset of R. Then λ is a L-fuzzy prime convex sub *l*-ring of R if and only if there exists a pair (P,α), where P is a prime convex *l*-sub ring and α is an irreducible element of L, such that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in P \\ \alpha, \text{otherwise} \end{cases}$$

Definition 3.19: A L-fuzzy sub *l*-ring λ of R is said to be a L-fuzzy *l*-ideal of R,

(i) if $x, a \in R$, $|x| \leq |a| \Rightarrow \lambda(x) \geq \lambda(a)$ and

(ii) $\lambda(xy) \geq \lambda(x) \vee \lambda(y)$ for all $x, y \in R$

As above, we can define L-fuzzy prime ideals and L-fuzzy maximal ideals.

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