FUZZY L-IDEALS IN L-RINGS

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Abstract: The aim of this paper is to introduce the notions of L-fuzzy *l*- ideals and L – Fuzzy α congruences of *l*-ring R with values in a complete lattice which is infinite meet distributive and investigate some properties and investigate the existence of one to one correspondence between the lattice L – Fuzzy *l*-ideals and the lattice of L – Fuzzy congruences of R; infact, this correspondence is a Lattice isomorphism.

Keywords: L-fuzzy *l*- ideals, L -Fuzzy prime *l*- ideals ,L-fuzzy maximal *l*- ideals and L – Fuzzy α Congruences.

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Introduction: Ever since L.A.Zadeh introduced the notion of fuzzy sets, the theory of fuzzy sets has attracted several researchers in the areas of Mathematics, Computer Science, Engineering and Technology. J.A.Goguen initiated a more abstract study of fuzzy sets by replacing the values set [0,1] by a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Most of the authors considered fuzzy subsets taking values in a complete lattice. Fuzzy algebra is now a well developed part of algebra. Partially ordered algebraic systems play an important role in algebra. Especially *l*-groups, *l*-rings, Vector lattices and f-rings are important concepts in algebra which present an abstract study of rings of continuous functions. In [13], we introduced L-fuzzy sub *l*-groups and L-fuzzy *l*-ideals. In [14], we introduced Fuzzy Convex sub *l*-groups and in [16], we studied L-fuzzy prime spectrum of *l*-groups. In [14], we introduced L – Fuzzy sub *l*-rings and L – Fuzzy Convex sub *l* –rings. The objective of this paper is to study L-fuzzy *l*- ideals of *l*-rings which assume values in a complete lattice which satisfies infinite meet distributive law.

In this paper, we introduce the concepts of L-fuzzy *l*- ideals, L -Fuzzy prime *l*- ideals and L-fuzzy maximal *l*- ideals, L-fuzzy α congruences of *l*-rings.

Throughout this paper, let $R \neq 0$ be an *l*-ring and L stands for a nontrivial complete lattice in which the infinite meet distributive law, $a \wedge (\vee_{s\in S} s) = \vee_{s\in S}(a \wedge s)$for any $S \subseteq L$ and $a \in L$ holds. Throughout the paper we consider meet irreducible elements of L only.

1. Preliminaries: Let $R = (R, +, \vee, \wedge)$ be an *l*-ring with 0 as the additive identity in R.
Definition 1.1: A L-fuzzy subset λ of R is said to be a L-fuzzy subring of R, if
i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$
ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$, for all $x,y \in R$.

Definition 1.2 : A L-fuzzy subset λ of R is said to be a L-fuzzy *l*- ideal of R, if
i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$
ii) $\lambda(xy) \geq \lambda(x) \vee \lambda(y)$, for all $x,y \in R$.

Definition 1.3 A L-fuzzy subset λ of R is said to be a L-fuzzy sub *l*-ring of R, if
i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$
ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$
iii) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda (y)$

iv) $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$ for all $x, y \in R$.

Definition 1.4: A L-fuzzy sub *l*-ring λ of R is said to be a L-fuzzy convex sub *l*-ring of R if $x, a \in G$, $0 \leq x \leq a \Rightarrow \lambda(x) \geq \lambda(a)$(Convexity condition).

2. L-Fuzzy *l*- ideals: In this section we introduce the concept of L-fuzzy *l*- ideals.

Definition 2.1: A L-fuzzy sub *l*-ring λ of R is said to be a L-fuzzy *l*-ideal of R,
(i) if $x, a \in R$, $|x| \leq |a| \Rightarrow \lambda(x) \geq \lambda(a)$ and
(ii) $\lambda(xy) \geq \lambda(x) \vee \lambda(y)$ for all $x, y \in R$

Theorem 2.2: A L-fuzzy subset λ of an *l*-ring R is a L-fuzzy *l*- ideal of R if and only if λ_t is an *l*-ideal of R for all $t \in \lambda(G) \cup \{t \in L / \lambda(0) \geq t\}$.

Theorem 2.3: If λ is a L-fuzzy *l*- ideal of R, then Supp(λ) is a ideal of R, if Supp $(\lambda) \neq \phi$ and L is regular. (i.e., if $a \neq 0$, $b \neq 0 \Rightarrow a \wedge b \neq 0$ where $a, b \in L$).

Theorem 2.4: If A is any *l*- ideal of R, $A \neq G$, then the L-fuzzy subset λ of R defined by

$$\lambda(x) = \begin{cases} s & \text{if } x \in A \\ t & \text{if } x \notin A, \end{cases}$$

where $s, t \in L$ and $t < s \neq 0$, is a L-fuzzy *l*- ideal of R.

Theorem 2.5: The intersection of any non empty family of L-fuzzy *l*- ideals of R is an *l*-ideal of R.

Theorem 2.6: Let λ be a L-fuzzy *l*- ideal of an *l*-ring R. Then $R_\lambda = \{x \in G / \lambda(x) = \lambda(0)\}$ is an *l*-ideal of R.

Definition 2.7: Let λ be a L-fuzzy subset of an *l*-ring of R. Let $\langle\lambda\rangle = \cap\{\mu \mid \lambda \subseteq \mu, \mu \text{ is any L} - \text{fuzzy sub } l - \text{ring of R}\}$. Then $\langle\lambda\rangle$ is called the L-fuzzy *l*- ideal of R generated by λ. Clearly $\langle\lambda\rangle$ is the smallest L-fuzzy sub *l*-ring of R which contains λ.

Theorem 2.8: Let μ be a L-fuzzy subset of an *l*-ring R. Define $\nu : R \rightarrow L$ be a L-fuzzy subset as follows: $\nu(x) = \vee\{\wedge_{y \in A}\mu(y) | A \subseteq R, 1 \leq |A| < \infty, x \in \langle A\rangle\}(x \in R)$.
Where $\langle A\rangle$ denotes *l*-ideal generated by A. Then $\nu = \langle\mu\rangle$, L-fuzzy *l*- ideal generated by μ.

Theorem 2.9: Let R and R^1 be two *l*-rings. Let λ and μ are two L-fuzzy *l*- ideals of R and R^1 respectively. If $f : R \rightarrow R^1$ be a homomorphism and onto then
(i) $f(\lambda)$ is a L-fuzzy *l*- ideal of R^1, provided that λ has sup property,
(ii) $f^{-1}(\mu)$ is a L-fuzzy *l*- ideal of R,
(iii) $(f(\lambda))(0^1) = \lambda(0)$, where $0^1 \in R^1$ and $0 \in R$,
(iv) $f(G_\lambda) \subseteq R^1_{f(\lambda)}$,
(v) If λ is constant on Ker f, then $(f(\lambda))(f(x)) = \lambda(x)$, for all $x \in R$,
(vi) $f^{-1}(R^1_\mu) = R_{f^{-1}(\mu)}$.
As an immediate consequence, if λ is constant on Ker f, it is easy to observe that
i) $f^{-1}(f(\lambda)) = \lambda$ and ii) $f(f^{-1}(\mu)) = \mu$.

3. L-fuzzy prime *l*- ideals and L – fuzzy maximal *l*-ideals: In this section we introduce L – Fuzzy prime *l*-ideals and L – Fuzzy maximal *l*-ideals and their characterizations.

Definition 3.1: Let λ be a L-fuzzy subset of an *l*-ring R. Then λ is called a L-fuzzy maximal *l*- ideal of R, if λ is a maximal element in the set of all non constant L-fuzzy *l*- ideals of R under point wise partial ordering.

Theorem 3.2: Let λ be a L-fuzzy subset of an *l*-ring R. Then λ is a L-fuzzy maximal *l*-ideal of R if and only if there exist, a maximal *l*- ideal M of R and maximal element α in L such that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in A \\ \alpha, \text{otherwise} \end{cases}$$

Definition 3.3: A non constant L-fuzzy convex sub *l*-ring of an *l*-ring R is called L-fuzzy prime *l*- ideal if and only if for any –fuzzy *l*- ideals μ and ν, $\mu \cap \nu \subseteq \lambda \Rightarrow$ either $\mu \subseteq \lambda$ or $\nu \subseteq \lambda$.

Lemma 3.4: If λ is a L-fuzzy prime *l*- ideal of R, then $\lambda(0) = 1$.

Theorem 3.5: Let λ be a L –fuzzy subset of R. Then λ is a L-fuzzy prime *l*- ideal of R if and only if there exists a pair (P, α), where P is a prime *l*- ideal and α is an irreducible element of L, such that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in P \\ \alpha, \text{otherwise} \end{cases}.$$

4. L – Fuzzy α - Congruences in *l*-rings:

In this section we discuss L – Fuzzy α - Congruences and one – to – one correspondence between the lattice L – Fuzzy *l*-ideals and the lattice of L – Fuzzy congruences of R.

Definition 4.1: Let $\alpha \in L - \{0\}$. Let ψ be a L – Fuzzy relation on R. ψ is called,

(i) α - reflexive : if $\psi(x, x) = \alpha$ and $\psi(x, y) \leq \alpha \; \forall \; x, \; y \in G$

(ii) Symmetric : if $\psi(x, y) = \psi(y, x)$, for all $x, \; y \in G$.

(iii) Transitive : if $\psi \, o\psi \subseteq \psi$, where $(\psi \, o\psi)(x, y) = \vee_{z \in R}[\psi(x, z) \wedge \psi(z, y)]$.

Definition 4.2: A L – Fuzzy relation ψ on R is called a L – fuzzy α - equivalence relation on R if ψ is (i) α - reflexive, (ii) Symmetric and (iii) Transitive.

Definition 4.3 : A L – fuzzy relation ψ is compatible on R if

$\psi(a + c, b + d) \geq \psi(a, b) \wedge \psi(c, d)$, $\psi(a \cdot c, b \cdot d) \geq \psi(a, b) \wedge \psi(c, d)$

$\psi(a \vee c, b \vee d) \geq \psi(a, b) \wedge \psi(c, d)$, $\psi(a \wedge c, b \wedge d) \geq \psi(a, b) \wedge \psi(c, d) \; \forall a, b, c, d \in R$.

Definition 4.4: A Compatible L – fuzzy α - equivalence relation on R is called a L-fuzzy α - congruence on R.

Lemma 4.5: If ψ is an L – fuzzy α - congruence on R, then $\psi(x, y) = \psi(-x, -y)$ for all $x, \; y \in G$.

Lemma 4.6: If ψ is a L – fuzzy α - congruence of R, then $\psi(x - y, 0) = \psi(x, y) \; \forall \; x, y \in G$.

Lemma 4.7: Intersection of any non empty family of L – fuzzy α - congruence relations on R, is a L – fuzzy α - congruence relation on R.

Theorem 4.8 : The set of all L- Fuzzy α - congruences $C(R, \alpha)$ is a complete lattice under the relation $\subseteq$ i.e., $(\theta, \psi \in C(R, \alpha), \theta \subseteq \psi \Leftrightarrow \theta(x, y) \leq \psi(x, y), \forall (x, y) \in R \times R)$.

Definition 4.9: Let μ be a L – fuzzy *l*- ideal of R such that $\mu(0) = \alpha$. A L – fuzzy relation θ_μ can be defined on R by

$$\theta_\mu(x, y) = \begin{cases} \mu(x - y) & \text{if } x \neq y \\ \alpha & \text{if } x = y \end{cases}.$$

Lemma 4.10: θ_μ is a L – fuzzy equivalence relation on R.

Lemma 4.11: $\theta_\mu(-x, -y) = \theta_\mu(x, y), \forall \; x, y \in R$.

Lemma 4.12: The L – fuzzy relation θ_μ is defined on R is L – Fuzzy compatible.

Theorem 4.13: θ_μ L – Fuzzy α - congruence on R.

Theorem 4.14: Let ψ be a L – Fuzzy α - congruence relation on R. Define the L – Fuzzy subset λ_ψ of R, by $\lambda_\psi(x) = \psi(x, 0), \forall x \in R$. Then λ_ψ is a L – fuzzy *l*- ideal of R.

Now, the following theorems gives a one to one correspondence between L – Fuzzy α - congruences and L – Fuzzy *l*- ideals of a *l*-ring R. We denote

$L_\alpha(R) = \{\mu \in L(R) | \mu(0) = \alpha\}$ and $C(R, \alpha)$ = Set of all L – Fuzzy α - congruences.

Theorem 4.15: If $\mu \in L_\alpha(R)$, then $\lambda_{(\theta_\mu)} = \mu$.

Theorem 4.16: If $\psi \in C(R, \alpha)$, then $\theta_{(\lambda_\psi)} = \psi$.

Theorem 4.17 : The mappings $\mu \rightarrow \psi_\mu : L_\alpha(R) \rightarrow C(R, \alpha)$ and $\theta \rightarrow \lambda_\theta : C(R, \alpha) \rightarrow L_\alpha(R)$ are mutual inverses. Moreover, the mappings are lattice isomorphisms.

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