INDUSTRIAL APPLICATIONS IN PHYSICAL & MATHEMATICAL SCIENCES

Editors
Dr.Ratnakar D B
Dr.K A Emmanuel
Dr.P Paul Divakar

INDUSTRIAL APPLICATIONS OF PHYSICAL & MATHEMATICAL SCIENCES

ISBN 978-81-944859-6-4

Editors

Dr.Ratnakar D B

Dr. K A Emmanuel

Dr. P. Paul Divakar

Compiled and Published by

SIR C R REDDY COLLEGE
(Aided& Autonomous), Eluru, A.P
Affiliated to Adikavi Nannaya University, Rajamahendravaram
[Thrice Accredited with A Grade by NAAC]
College with Potential for Excellence
An ISO 9001:2015 Certified Institution

In collaboration with

IMRF INSTITUTE OF HIGHER EDUCATION & RESEARCH, INDIA
www.imrfedu.org

CONTENTS

 | ISBN 978-81-944859-6-4

Chapter: 1

FUZZY L-IDEALS IN L-RINGS

Dr. G.S.V.Satya Saibaba

Introduction: Ever since L.A.Zadeh introduced the notion of fuzzy sets, the theory of fuzzy sets has attracted several researchers in the areas of Mathematics, Computer Science, Engineering and Technology. J.A.Goguen initiated a more abstract study of fuzzy sets by replacing the values set [0,1] by a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Most of the authors considered fuzzy subsets taking values in a complete lattice. Fuzzy algebra is now a well developed part of algebra. Partially ordered algebraic systems play an important role in algebra. Especially *l*-groups, *l*-rings, Vector lattices and f-rings are important concepts in algebra which present an abstract study of rings of continuous functions. In [13], we introduced L-fuzzy sub *l*-groups and L-fuzzy *l*-ideals. In [14], we introduced Fuzzy Convex sub *l*-groups and in [16], we studied L-fuzzy prime spectrum of *l*-groups. In [14], we introduced L – Fuzzy sub *l*-rings and L – Fuzzy Convex sub *l* –rings. The objective of this paper is to study L-fuzzy *l*- ideals of *l*-rings which assume values in a complete lattice which satisfies infinite meet distributive law.

In this paper, we introduce the concepts of L-fuzzy *l*- ideals, L -Fuzzy prime *l*- ideals and L-fuzzy maximal *l*- ideals, L-fuzzy α congruences of *l*-rings.

Throughout this paper, let $R \neq 0$ be an *l*-ring and L stands for a nontrivial complete lattice in which the infinite meet distributive law, $a \wedge (\vee_{s\in S} s) = \vee_{s\in S}(a \wedge s)$ for any $S \subseteq L$ and $a \in L$ holds. Throughout the paper we consider meet irreducible elements of L only.

1. Preliminaries: Let $R = (R, +, \vee, \wedge)$ be an *l*-ring with 0 as the additive identity in R.

Definition 1.1: A L-fuzzy subset λ of R is said to be a L-fuzzy subring of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$, for all $x,y \in R$.

Definition 1.2 : A L-fuzzy subset λ of R is said to be a L-fuzzy *l*- ideal of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \vee \lambda(y)$, for all $x,y \in R$.

Definition 1.3 A L-fuzzy subset λ of R is said to be a L-fuzzy sub *l*-ring of R, if

i) $\lambda(x - y) \geq \lambda(x) \wedge \lambda(y)$

ii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$

iii) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda (y)$

iv) $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda (y)$ for all $x,y\in R$.

Definition 1.4: A L-fuzzy sub *l*-ring λ of R is said to be a L-fuzzy convex sub *l*-ring of R x, $a \in G$, $0 \leq x \leq a \Rightarrow \lambda(x) \geq \lambda(a)$(Convexity condition).

2. L-Fuzzy *l*- ideals: In this section we introduce the concept of L-fuzzy *l*- ideals.

Definition 2.1: A L-fuzzy sub *l*-ring λ of R is said to be a L-fuzzy *l*-ideal of R,

(i) if $x, a \in R$, $|x| \leq |a| \Rightarrow \lambda(x) \geq \lambda(a)$ and

(ii) $\lambda(xy) \geq \lambda(x) \vee \lambda(y)$ for all $x, y \in R$

Theorem 2.2: A L-fuzzy subset λ of an *l*-ring R is a L-fuzzy *l*- ideal of R if and only if λ_t is an ideal of R for all $t \in \lambda (G) \cup \{t \in L / \lambda(o) \geq t\}$.

Theorem 2.3: If λ is a L-fuzzy *l*- ideal of R, then Supp(λ) is a ideal of R, if Supp (λ) $\neq \phi$ and L regular. (i.e., if $a \neq o$, $b \neq o \Rightarrow a \wedge b \neq o$ where $a, b \in L$).

Theorem 2.4: If A is any *l*- ideal of R, $A \neq G$, then the L-fuzzy subset λ of R defined by

$$\lambda(x) = \begin{cases} s & \text{if } x \in A \\ t & \text{if } x \notin A, \end{cases}$$

where $s, t \in L$ and $t < s \neq o$, is a L-fuzzy *l*- ideal of R.

Theorem 2.5: The intersection of any non empty family of L-fuzzy *l*- ideals of R is an *l*-ideal R.

Theorem 2.6: Let λ be a L-fuzzy *l*- ideal of an *l*-ring R. Then $R_\lambda = \{x \in G/ \lambda(x) = \lambda(0)\}$ is an ideal of R.

Definition 2.7: Let λ be a L-fuzzy subset of an l-ring of R. Let $\langle\lambda\rangle = \cap\{\mu \mid \lambda \subseteq \mu, \mu$ is any L – fuzzy sub l – ring of R $\}$. Then $\langle\lambda\rangle$ is called the L-fuzzy l- ideal of R generated by λ. Clearly $\langle\lambda\rangle$ is the smallest L-fuzzy sub l-ring of R which contains λ.

Theorem 2.8: Let μ be a L-fuzzy subset of an l-ring R. Define $\nu : R \to L$ be a L-fuzzy subset as follows: $\nu(x) = \vee\{\wedge_{y\in A}\mu(y) \mid A \subseteq R, 1 \leq |A| < \infty, x \in \langle A\rangle\}(x \in R)$.

Where $\langle A\rangle$ denotes l-ideal generated by A. Then $\nu = \langle\mu\rangle$, L-fuzzy l- ideal generated by μ.

Theorem 2.9: Let R and R' be two l-rings. Let λ and μ are two L-fuzzy l- ideals of R and R' respectively. If $f : R \to R'$ be a homomorphism and onto then

(i) $f(\lambda)$ is a L-fuzzy l- ideal of R', provided that λ has sup property,

(ii) $f^{-1}(\mu)$ is a L-fuzzy l- ideal of R,

(iii) $(f(\lambda))(o') = \lambda(o)$, where $o' \in R'$ and $o \in R$,

(iv) $f(G_\lambda) \subseteq R'_{f(\lambda)}$,

(v) If λ is constant on Ker f, then $(f(\lambda))(f(x)) = \lambda(x)$, for all $x \in R$,

(vi) $f^{-1}(R^1_\mu) = R_{f^{-1}(\mu)}$.

As an immediate consequence, if λ is constant on Ker f, it is easy to observe that
i) $f^{-1}(f(\lambda)) = \lambda$ and ii) $f(f^{-1}(\mu)) = \mu$.

3. L-fuzzy prime l- ideals and L – fuzzy maximal l-ideals: In this section we introduce L - Fuzzy prime l-ideals and L – Fuzzy maximal l-ideals and their characterizations.

Definition 3.1: Let λ be a L-fuzzy subset of an l-ring R. Then λ is called a L-fuzzy maximal l- ideal of R, if λ is a maximal element in the set of all non constant L-fuzzy l- ideals of R under point wise partial ordering.

Theorem 3.2: Let λ be a L-fuzzy subset of an l-ring R. Then λ is a L-fuzzy maximal l-ideal of R if and only if there exist, a maximal l- ideal M of R and maximal element α in L such that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in A \\ \alpha, \text{otherwise} \end{cases}$$

Definition 3.3: A non constant L-fuzzy convex sub l-ring of an l-ring R is called L-fuzzy prime l- ideal if and only if for any –fuzzy l- ideals μ and ν, $\mu \cap \nu \subseteq \lambda \Rightarrow$ either $\mu \subseteq \lambda$ or $\nu \subseteq \lambda$.

Lemma 3.4: If λ is a L-fuzzy prime l- ideal of R, then $\lambda(o) = 1$.

Theorem 3.5: Let λ be a L –fuzzy subset of R. Then λ is a L-fuzzy prime l- ideal of R if and if there exists a pair (P, α), where P is a prime l- ideal and α is an irreducible element of L, that

$$\lambda(x) = \begin{cases} 1, \text{if } x \in P \\ \alpha, \text{otherwise} \end{cases}.$$

4. L – Fuzzy α - Congruences in *l*-rings:

In this section we discuss L – Fuzzy α - Congruences and one – to – one correspondence between the lattice L – Fuzzy *l*-ideals and the lattice of L – Fuzzy congruences of R.

Definition 4.1: Let $\alpha \in L - \{0\}$. Let ψ be a L – Fuzzy relation on R. ψ is called,

(i) α - reflexive : if $\psi(x,x) = \alpha$ and $\psi(x,y) \le \alpha \; \forall \; x, \; y \in G$

(ii) Symmetric : if $\psi(x,y) = \psi(y,x)$, for all $x, \; y \in G$.

(iii) Transitive : if $\psi \, o \psi \subseteq \psi$, where $(\psi \, o \psi)(x,y) = \vee_{z \in R}[\psi(x,z) \wedge \psi(z,y)]$.

Definition 4.2: A L – Fuzzy relation ψ on R is called a L – fuzzy α - equivalence relation on R if ψ is (i) α - reflexive, (ii) Symmetric and (iii) Transitive.

Definition 4.3 : A L – fuzzy relation ψ is compatible on R if

$\psi(a + c, b + d) \ge \psi(a, b) \wedge \psi(c, d)$, $\psi(a \cdot c, b \cdot d) \ge \psi(a, b) \wedge \psi(c, d)$

$\psi(a \vee c, b \vee d) \ge \psi(a, b) \wedge \psi(c, d)$, $\psi(a \wedge c, b \wedge d) \ge \psi(a, b) \wedge \psi(c, d) \; \forall a, b, c, d \in R$.

Definition 4.4: A Compatible L – fuzzy α - equivalence relation on R is called a L-fuzzy α - congruence on R.

Lemma 4.5: If ψ is an L – fuzzy α - congruence on R, then $\psi(x,y) = \psi(-x,-y)$ for all $x, \; y \in G$.

Lemma 4.6: If ψ is a L – fuzzy α - congruence of R, then $\psi(x - y, 0) = \psi(x,y) \; \forall \; x, y \in G$.

Lemma 4.7: Intersection of any non empty family of L – fuzzy α - congruence relations on R, is a L –fuzzy α - congruence relation on R.

Theorem 4.8 : The set of all L- Fuzzy α - congruences $C(R, \alpha)$ is a complete lattice under the relation $\subseteq$ i.e., $(\theta, \psi \in C(R, \alpha), \theta \subseteq \psi \Leftrightarrow \theta(x,y) \le \psi(x,y), \forall (x,y) \in R \times R)$.

Definition 4.9: Let μ be a L – fuzzy *l*- ideal of R such that $\mu(0) = \alpha$. A L – fuzzy relation θ_μ can be defined on R by

$$\theta_\mu(x,y) = \begin{cases} \mu(x - y) & \text{if } x \ne y \\ \alpha & \text{if } x = y \end{cases}.$$

Lemma 4.10: θ_μ is a L – fuzzy equivalence relation on R.

Lemma 4.11: $\theta_\mu(-x,-y) = \theta_\mu(x,y), \forall \; x, y \in R$.

Lemma 4.12: The L – fuzzy relation θ_μ is defined on R is L – Fuzzy compatible.

Theorem 4.13: θ_μ L – Fuzzy α - congruence on R.

Theorem 4.14: Let ψ be a L – Fuzzy α - congruence relation on R. Define the L – Fuzzy subset λ_ψ, of R, by $\lambda_\psi(x) = \psi(x, 0), \forall x \in R$. Then λ_ψ, is a L – fuzzy *l*- ideal of R.

Now, the following theorems gives a one to one correspondence between L – Fuzzy α - congruences and L – Fuzzy *l*- ideals of a *l*-ring R. We denote

$L_\alpha(R) = \{\mu \in L(R) | \mu(0) = \alpha\}$ and $C(R, \alpha)$ = Set of all L – Fuzzy α - congruences.

Theorem 4.15: If $\mu \in L_\alpha(R)$, then $\lambda_{(\theta_\mu)} = \mu$.

Theorem 4.16: If $\psi \in C(R, \alpha)$, then $\theta_{(\lambda_\psi)} = \psi$.

Theorem 4.17 : The mappings $\mu \to \psi_\mu : L_\alpha(R) \to C(R, \alpha)$ and $\theta \to \lambda_\theta : C(R, \alpha) \to L_\alpha(R)$ are mutual inverses. Moreover, the mappings are lattice isomorphisms.

References:

1. Dixit, V.N., Kumar, R and Ajmal, N., Fuzzy *l*- ideals and fuzzy prime ideals of ring, Fuzzy sets and systems, 44 (1991) 127-138.
2. Fuchs, L., Partially ordered algebraic systems, Pergamon Press, 1963.
3. Garrett Birkhoff, Lattice Theory, Americal Mathematical Society Colloquim publications Volume XXV.
4. Goguen, J.A., L-fuzzy sets, J.Math. Anal.Appl. 18 (1967), 145 -174.
5. Keimel, K., The representation of lattice ordered groups and rings, by sections in sheave Lecture notes in Maht., Vol.248, Springer, Berling, 1971.
6. Kumar, R., Fuzzy irreducible ideals in rings, Fuzzy sets and systems 42 (1991) 360-379.
7. Kumar, R., Fuzzy Algebra, Volume 1, University of Delhi Publication Division, 1993.
8. Mordeson, J.N and Malik, D.S., Fuzzy commutative algebra, World Scientific publishir Co. pvt. Ltd.
9. Naseem Ajmal, Fuzzy lattices, information Sciences 79 (1994), 271 – 291.
10. Rosenfeld, A., fuzzy groups, J.Math. Anal. Appl. 35 (1971) , 512-217.
11. Saibaba G.S.V.S., Fuzzy lattice ordered goups, Southeast Asian Bulletin of Mathematics (2008), 749-766.
12. Saibaba G.S.V.S., Fuzzy convex sub *l*-groups, Annals of fuzzy Mathematics and informa 11 (6) 2016, b989-1001.
13. Saibaba G.S.V.S., L-fuzzy Prime Spectrum of *l*-groups, Annals of fuzzy Mathematics informatics 12(2)2016, 175-191.
14. Saibaba G.S.V.S., Fuzzy convex sub *l*- rings, Mathematical Sciences International Rese Journal, Vol.8, issue 2, Nov-2019, 119 – 123.
15. Swamy, U.M. and Viswanadha Raju, D., Algebraic fuzzy systems, Fuzzy sets and syst 41(1991), 187-194.
16. Swamy, U.M. and Viswanadha Raju, D.,Irreducibility in algebraic fuzzy systems, Fuzz and systems, 41(1991), 233-241.
17. Zadeh, L.A., Fuzzy sets, Inform and control, 8 (1965), 338-353.